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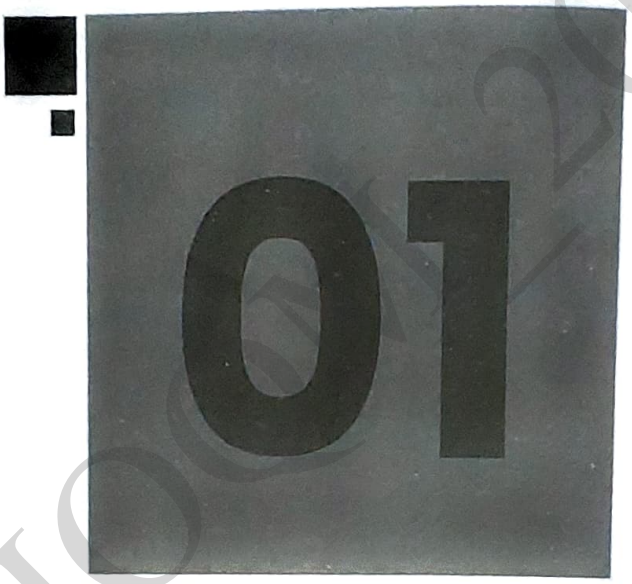

Number Theory

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Chapter 01

Number Theory

1. Number Theory

Number theory is a branch of pure mathematics devoted to the study of integers and their properties. Number theory was previously known as arithmetic. In number theory, most of the time, we deal with natural numbers. The natural numbers are classified into different types, such as odd numbers, even numbers, prime numbers etc. We will encounter all these types of numbers, and many others, in our excursion through the number theory.

1.1 Basics of Number Theory

Prime and Composite Numbers

Definition: An integer $p > 1$ is called a prime number, or a prime, if it has no divisor d such that $1 < d < p$. If an integer is not a prime, then it is called a composite number.

Note that 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ... are prime numbers, while 4, 6, 8, 15, 20, ... are composite numbers.

Fundamental Theorem of Arithmetic

Any natural number different from 1 can be uniquely represented as product of prime numbers in the increasing order.

Remark: Fundamental Theorem of Arithmetic implies that every integer $n > 1$ can be written as $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, where $p_i \neq p_j$ whenever $i \neq j$. In fact, one may assume that $p_i < p_j$ whenever $i < j$, where p_i are distinct primes.

Ex: $270 = 2 \times 3^3 \times 5$.

Co-Prime Numbers

Two natural numbers are called relatively prime, or co-prime, if they have no common divisors greater than 1.

For example: 9 and 16 are relatively prime integers. Similarly, 8 and 15 are co-prime, while 6 and 15 are not.

Note: Any two consecutive integers are relatively prime. Further, any two consecutive odd integers are relatively prime.

Divisibility of Integers

An integer $a \neq 0$ divides b, if there exists an integer m such that $b = am$, and thus, we write as $a \mid b$ (read as 'a divides b'). This can also be stated as 'b is divisible by a' or 'a is a divisor of b' or 'b is a multiple of a'.

Important Results:

Let a, b, c, m, x, y be integers

(i) If $a|b$ then $a|bc$ for any integer c

(ii) If $a|b$ and $b|c$ then $a|c$

(iii) If $a|b$ and $a|c$ then $a|(bx + cy)$ for any integers x and y

(iv) If $a|b$, and $b \neq 0$ then $|a| \leq |b|$

(v) If $a|b$ and $b|a$ then $a = \pm b$

(vi) If $m \neq 0$ then $a|b$ if and only if $ma|mb$

Proof:

(i) If $a|b$, then $b = am$ where m is an integer. Hence $bc = a(mc)$ for any integer c. Hence, $a|bc$.

(ii) If $a|b$ and $b|c$, then $b = am$ and $c = bm_1$ where m, m_1 are integers. Thus $c = (am)m_1 = a(mm_1)$. Hence, $a|c$.

(iii) If $a|b$ and $a|c$, then $b = am$, $c = am_1$ where $m, m_1 \in Z$. Hence, $bx + cy = a(mx + m_1y)$. Hence, $a|(bx + cy)$.

(iv) If $a|b$, $b \neq 0$, then $b = am$, $m \neq 0$. Hence, $|b| = |am| = |a||m|$. As $m \neq 0$, $|m| \geq 1$, hence $|a| \leq |b|$.

(v) If $a|b$ then $|a| \leq |b|$. Also, $b|a$ implies $|b| \leq |a|$. Thus, $|a| = |b|$. Hence, $a = \pm b$.

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(vi) If a|b, then b = an. Suppose n ≠ 0. Then mb = (ma)n. Hence ma|mb. Conversely, if ma|mb, then mb = man. Since, m ≠ 0, we get b = an, that is a|b.

Note: If a composite number is represented as $n = p_1^{\alpha_1} \times p_1^{\alpha_2} \times ..\times p_m^{\alpha_m}$, where $p_1, p_2, ..., p_m$ are prime numbers, then the number of positive divisors of n is $(\alpha_1 + 1)(\alpha_2 + 1).....(\alpha_m + 1)$. For example, number of divisors of $n = 2^3 3^4$ are $(3 + 1)(4 + 1) = 20$

Factorial of a Number(n!)

Factorial of a number n is the product of all positive integers less than or equal to that number. It is denoted by symbol n!.
For a positive integer n, $n! = n \times (n - 1) \times (n - 2) \times ..... \times 2 \times 1$
For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Note: $0! = 1$

Exponent of a Prime in Factorial

Exponent of any prime p in n! means what is the power of prime p in the prime factorization of n!.
For example, in $5! = 2^3 \times 3 \times 5$ the exponent of 2 is three, exponent of 3 is one and exponent of 5 is also one.

Exponent of any prime p in n! is given by $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+....$

Here [x] represent greatest integer less than or equal to x.

For example, exponent of 5 in 29! = $\left[\frac{29}{5}\right]+\left[\frac{29}{5^2}\right] = 5 + 1 = 6$, therefore 29! contain 5^6.

Example 1: Is $2^9.3$ divisible by 6?

Sol. Yes, since 6 = 2.3, and the prime factorization of the given number contains both the prime numbers 2 and 3

Example 2: Is it true that if a natural number is divisible by 4 and 3, then it must be divisible by $4 \times 3 = 12$?

Sol. Yes. Indeed, the prime factorization of a natural number which is divisible by 4 must contain at least two 2's. Since this number is also divisible by 3, there is at least one 3. Therefore, our number is divisible by $3 \times 2 \times 2 = 12$.

Example 3: The number A is not divisible by 3. Is it possible that the number 2A is divisible by 3?

Sol. No, since 3 does not belong to the prime factorization of A, and, therefore, 3 does not belong to the prime factorization of 2A

Example 4: What is the largest prime factor of 143000000?

Sol. $143000000 = 11 \times 13 \times 2^6 \times 5^6$
Largest prime factor = 13

Example 5: Three positive integers are each greater than 1, have a product of 27000, and are pairwise relatively prime. What is their sum?

Sol. Prime factorization of $27000 = 3^3 2^3 5^3$
Clearly, 3^3, 2^3 and 5^3 are pairwise prime.
$3^3 + 2^3 + 5^3 = 27 + 8 + 125 = 160$

Example 6: When 270 is divided by the odd number x, the quotient is a positive prime and the remainder is 0. What is the value of x?

Sol. $270 = 3^3 \times 2^1 \times 5^1$
Odd number x can be 5, 3, 3^2, 3^3, 5×3, 5×3^2, 5×3^3

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In case of $x = 5 \times 3^1$, we will get a prime quotient $\frac{270}{3^1 \times 5^1} = 2$

$x = 135$

Example 7: What is the smallest positive integer that is neither prime nor square and that has no prime factor less than 50?

Sol. Primes greater than 50 are 53, 59, 61,..... Number is not prime, if it contains more than one prime factor or some power of a single prime. As per the question smallest number will be 53×59.

Example 8: Find the smallest natural number n such that n! is divisible by 990.

Sol. $990 = 11 \times 3 \times 2 \times 5$

n! must contain $11 \Rightarrow n = 11$

Example 9: For some number n, can the number n! have exactly five zeros at the end of its decimal representation?

Sol. No, for five zeros, n! must contain $2^5 5^5$

24! contains $2^{22} 5^4$ and 25! contains $2^{22} 5^6$

24! contains four zeros and 25! contains six zeros.

So, no such n is possible.

Note: Exponent of 2 is always more than exponent of 5 in n! ($n > 1$). Therefore, while finding the number of zeros at the end of n!, just find the exponent of 5 in n!

Example 10: Find all solutions in natural numbers of the equations $x^2 - y^2 = 31$.

Sol. $(x - y)(x + y) = 31$

$(x - y)(x + y) = 31 \times 1$

$x - y = 1 \ \& \ x + y = 31$

$(\because x + y > x - y)$

$\Rightarrow x = 16, y = 15$

GCD and LCM

The Greatest Common Divisor (GCD) OR Highest Common Factor (HCF)

GCD of two natural numbers is the greatest natural number which divides them both.

If d is GCD of a and b then

(i) d|a and d|b

(ii) If c | a and c | b, then c | d

We write it as (a, b) = d

Prime factorization method to find GCD

GCD of two numbers is the product of common prime factors raised to their lowest powers.

For example, gcd of $2^5.3^4.7^3.11^5$ and $2^2.5^3.7^1.11^2 = 2^2.7^1.11^2$

Properties of GCD

Generally (a, b) represents GCD of a and b. Here a, b, c, d and m are positive integers.

1. If (b, c) = d and h is a common divisor of b and c, then h is a divisor of d.
2. For any $m > 0$, (mb, mc) = m (b, c)
3. If d|b and d|c and $d > 0$, then $\left(\frac{b}{d}, \frac{c}{d}\right) = \left(\frac{1}{d}\right)(b,c)$
4. If (b, c) = d, then $\left(\frac{b}{d}, \frac{c}{d}\right) = \left(\frac{1}{d}\right)(b,c)$
5. If (a, b) = 1 and (a, c) = 1 then (a, bc) = 1
6. If a | bc and (a, b) = 1, then a | c
7. If a | b, then (a, b) = a

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Example 11: Find gcd(108, 72)

Sol. $108 = 3^3 \times 2^2, 72 = 2^3 \times 3^2$

$\gcd(108, 72) = \gcd(3^3 \times 2^2, 2^3 \times 3^2) = 2^2 \times 3^2 = 36$

Note: GCD of two coprime numbers is 1. For example, gcd(5, 12) = 1

Example 12: Given numbers $A = 2^3.3^{10}.5.7^2$ and $B = 2^5.3.11$ find gcd (A, B).

Sol. $\gcd(A, B) = 2^3.3$

The Least Common Multiple (L.C.M.) of two natural numbers is the least natural number which is divisible by both of them.

LCM of two numbers x and y is denoted by lcm(x, y)

Prime factorization method to find LCM:

LCM of two numbers is the product of all prime factors raised to their highest powers.

For example, lcm of $2^5 3^4 7^3 11^5$ and $2^2 5^3 7^1 11^2 = 2^5 3^4 5^3 7^3 11^5$

Example 13: Find lcm(108, 72)

Sol. $\text{lcm}(108, 72) = \text{lcm}(3^3 \times 2^2, 2^3 \times 3^2) = 2^3 \times 3^3 = 216$

Note: lcm of two coprime numbers is equal to product of the numbers. For example, lcm(5, 12) = 60

Example 14: Prove that any two natural numbers a and b satisfy the equation $\gcd(a, b) \times \text{lcm}(a, b) = ab$.

Sol. Let gcd(a, b) = h

a = hm, b = hn, here m and n are coprime.

lcm(a, b) = lcm(hm, hn) = hmn

$\gcd(a, b) \times \text{lcm}(a, b) = h \times hmn = hm \times hn = ab$

Note: Remember this as a result. $\gcd(a, b) \times \text{lcm}(a, b) = ab$.

Example 15: Given the number $A = 2^8.5^3.7$ and $B = 2^5.3.5^7$ find lcm (A, B).

Sol. lcm (A, B) = $2^8.3.5^7.7$

Example 16: What is the greatest possible sum of two multiples of 12 each less than 100, whose greatest common factor is 24?

Sol. As GCD = 24, both the numbers are multiple of 24. Clearly 96 & 72 follow these properties. 96 + 72 = 168

Exercise - 1

1. The number 5A is divisible by 3. Is it true that A must be divisible by 3?
2. The number 15A is divisible by 6. Is it true that A must be divisible by 6?
3. Which of the following numbers is a perfect square?

   (a) $\frac{14!15!}{2}$ (b) $\frac{15!16!}{2}$ (c) $\frac{16!17!}{2}$ (d) $\frac{17!18!}{2}$
4. A standard six-sided die is rolled, and P is the product of the five numbers that are visible. What is the largest number that is certain to divide P?
5. The product of any two of the positive integers 30, 72 and N is divisible by the third. What is the smallest possible value of N?
6. Find all solutions in natural numbers of the equations: $x^2 - y^2 = 303$.
7. Let (a, b) denote the greatest common factor of a and b, and let [a, b] denote the least common multiple of a and b. Evaluate the following expressions

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(a) (15,20) (b) (66,121) (c) $(12^6, 18^4)$ (d) (2143567, 2143369)

8. Given a prime number P, find the number of natural numbers which are
   (a) Less than P and relatively prime to it (b) Less than P^2 and relatively prime to it.
9. The numbers a and b satisfy the equation 56a = 65b. Prove that a + b is composite.
10. Find all natural numbers which become 9 times greater if you insert zero between their units digit and their tens digit.

1.2 Division Algorithm

Given any positive integers a and b with $a \neq 0$, there exist unique non - negative integers q and r such that $b = qa + r$, $0 \leq r < a$. If $a \nmid b$, then r satisfies the stronger inequality $0 < r < |a|$. Here q: Quotient, r: Remainder, a: Divisor, b: Dividend. For example, if a = 2, then r can be 0 or 1. So any number when divided by 2 can be of the form 2q or 2q + 1. Similarly, if a = 3, then r can be 0, 1 or 2. So any number when divided by 3 can be of the form 3q, 3q + 1 or 3q + 2(which can also be written as 3q - 1) & so on.

Note: Any prime number greater than 3 is either 6k + 1 or 6k + 5 form.

Example 1: Show that the square of any integer is of the form 4k or 8k + 1. Here $k \in Z$.

Sol. Any number n can be written as
n = 4m, 4m + 1, 4m + 2, 4m+3

(i) $n = 4m$, $n^2 = 16m^2 = 4(4m^2) = 4p$

(ii) $n = 4m + 1$
$n^2 = 16m^2 + 8m + 1 = 4(4m^2 + 2m) + 1$
$n^2 = 4q + 1$

(iii) $n = 4m + 2$
$n^2 = 16m^2 + 4 + 16m = 4(4m^2 + 1 + 4m)$
$n^2 = 4r$

(iv) $n = 4m+3$
$n^2 = 16m^2 + 9 + 24m = 4(4m^2 + 2 + 6m) + 1$
$n^2 = 4s + 1$

Example 2: Show that square of an integer is of the form 9m or 3m + 1.

Sol. Any number n can be written as
n = 3k, 3k+1, 3k+2

(i) $n = 3k$
$n^2 = 9k^2 = 9m_1$

(ii) $n = 3k+1$
$n^2 = 9k^2 + 1 + 6k = 3m_2 + 1$

(iii) $n = 3k + 2$
$n^2 = 9k^2 + 4 + 12k = 3m_3 + 1$

Example 3: Show that cube of an integer is of the form 7m, 7m + 1 or 7m – 1.

Sol. Any number n can be written as
n = 7k, 7k + 1, 7k + 2, 7k + 3, 7k + 4, 7k + 5, 7k + 6

(i) $n = 7k$
$n^3 = 7m$

(ii) $n = 7k + 1$
$n^3 = (7k)^3 + 1 + 3 \times 7k \times 1\,(7k + 1) = 7m + 1$

(iii) $n = 7k + 2$

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$n^3 = (7k)^3 + 0 + 3 \times 7k \times 2(7k + 2) = 7m + 1$

Similarly for

(iv) $n = 7k + 3$, $n^3 = 7m - 1$

(v) $n = 7k + 4$, $n^3 = 7m + 1$,

(vi) $n = 7k + 5$, $n^3 = 7m - 1$,

(vii) $n = 7k + 6$, $n^3 = 7m - 1$.

Example 4: Show that any prime p is of the form $6k + 1$ or $6k + 5$ for $p > 3$.

Sol. As $p > 3$ so prime number must be odd number

Any number n can be written as

$n = 6k, 6k + 1, 6k + 2, 6k + 3, 6k + 4, 6k + 5$

As we can see $6k, 6k + 2, 6k + 3, 6k + 4$ are not prime as they are divisible by 6, 2, 3, 2 respectively so it may be a prime.

So, any prime p will be of the form $6k + 1$ or $6k + 5$

Note: Remember this as a result. Prime numbers greater than 3 can be written as $6k + 1$ or $6k + 5$.

Example 5: Find all integers n such that $n^2 + 1$ is divisible by $n + 1$.

Sol. $\frac{(n^2 + 1)}{n + 1}$

$= \frac{(n^2 - 1) + 2}{n + 1}$

$= (n - 1) + \frac{2}{n + 1}$ for this to be an interior, $n + 1$ divides 2

$\Rightarrow n = 0, 1, -2, -3$

Example 6: If p is a prime greater than 3 then show that $2p + 1$ and $4p + 1$ cannot be primes simultaneously.

Sol. Since p is a prime greater than 3, p is either of the type $6k + 1$ or $6k + 5$.

(i) If p is of the type $6k + 1$, then $2p + 1 = 2(6k + 1) + 1$

$= 12k + 3 = 3(4k + 1)$

$\Rightarrow 2p + 1$ cannot be a prime.

(ii) If p is of the type $6k + 5$ then $4p + 1 = 4(6k + 5) + 1 = 3(8k + 7)$

$\Rightarrow 4p + 1$ cannot be a prime.

Example 7: Prove that the product of any five consecutive natural numbers is divisible by 120.

Sol. $m = n(n + 1)(n + 2)(n + 3)(n + 4)$

Any number n can be written as

$n = 5k, 5k+1, 5k+2, 5k+3, 5k+4$

(i) if $n = 5k+1$

$m = (5k+1)(5k+2)(5k+3)(5k+4)(5k+5)$

$\Rightarrow$ m is multiple of 5

Similarly in other cases also m is multiple of 5

In five consecutive numbers there is at least one multiple of 4 & two multiple of 2. Therefore, m is divisible

Similarly, there is atleast one multiple of 3 in five consecutive numbers. Hence number is divisible by 120.

Note: Product of n consecutive integers is always divisible by n!.

Example 8: Prove that the number $n^3 + 2n$ is divisible by 3 for any natural number n.

Sol. n can be written as $3k$, $3k + 1$ or $3k + 2$

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Case (i) When $n = 3k$

$n^3 + 2n = n(n^2 + 2) = 3k(9k^2 + 2) = 3m$

Case (ii) When $n = 3k + 1$

$n^3 + 2n = n(n^2 + 2) = (3k + 1)(9k^2 + 6k + 3) = 3m$

Case (iii) When $n = 3k + 2$

$n^3 + 2n = n(n^2 + 2) = (3k + 2)(9k^2 + 12k + 6) = 3m$

Therefore $n^3 + 2n$ is divisible by 3 for any natural number n

Example 9: Prove that $n^2 + 1$ is not divisible by 3 for any integer n

Sol. n can be 3k, 3k + 1 or 3k + 2

(i) $n = 3k$

$n^2 + 1 = 9k^2 + 1 = 3m + 1$

(ii) $n = 3k + 1$

$n^2 = 9k^2 + 6k + 2 = 3m + 2$

(iii) $n = 3k + 2$

$n^2 + 1 = 9k^2 + 12k + 5 = 3m + 2$

Example 10: Prove that $n^3 - n$ is divisible by 24 for any odd n.

Sol. $n^3 - n = (n - 1)n(n + 1)$

As n is odd so n can be 4k + 1 or 4k + 3

(i) If $n = 4k + 1$

$n^3 - n = (4k)\ (4k+1)\ (4k+2) = 8m$

(ii) If $n = 4k + 3$

$n^3 - n = (4k + 2)\ (4k + 3)(4k + 4) = 8m$

In three consecutive integers there is at least one multiple of 3

Therefore, this number is divisible by 24.

Example 11: Prove that $p^2 - 1$ is divisible by 24 if p is a prime number greater than 3.

Sol. Any prime number greater than 3, can be written as

$p = 4k + 1$ or $4k + 3$

(i) $p = 4k + 1$

$p^2 - 1 = (4k)(4k + 2) = 8m$

(ii) $p = 4k+3$

$p^2 - 1 = (4k + 2)(4k + 4) = 8m$

p can also be written as 3k + 1 or 3k + 2

(i) If $p = 3k + 1$ then $(p^2 - 1) = (3k)(3k + 2) = 3m$

(ii) If $p=3k + 2$ then $(p^2 - 1) = (3k + 1)(3k + 3) = 3m \Rightarrow p^2 - 1$ multiple of 24.

Exercise – 2

1. Given natural numbers a, b, and c such that a +b+ c is divisible by 6, prove that $a^3 + b^3 + c^3$ is also divisible by 6.
2. Prove that $n^3 + 2$ is not divisible by 9 for any integer n.
3. The natural numbers x, y, and z satisfy the equation $x^2 + y^2 = z^2$. Prove that at least one of them is divisible by 3.
4. Prove that $n^5 + 4n$ is divisible by 5 for any integer n.
5. (a) Given that p, p + 10, and p + 14 are prime numbers, find p.
   (b) Given that p, 2p + 1, and 4p + 1 are prime numbers, find p.
6. Given the pair of prime numbers p and $p^2 + 2$, prove that $p^3 + 2$ is also a prime number.
7. Prove that the sum of the squares of five consecutive natural numbers cannot be a perfect square.

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8. Prove that $p^2 - q^2$ is divisible by 24 if p and q are prime numbers greater than 3.
9. Find all integers which leave remainder 1 when divided by 3, remainder 2 when divided by 10.
10. Find all primes p such that both p and $p^2 + 8$ are primes.

1.3 Unit Digit & Last Two Digits of a Number

In this section, we will see the method to find the unit digit and last two digits of numbers of the form a^n, where a and n are positive integers.

Cyclicity of a Number:

The cyclicity of any number primarily focuses on its unit digit. Each unit digit has its own repeating pattern when raised to any power. The concept of cyclicity of numbers can be learned by finding the unit digits of all single digit numbers from 0 to 9 when raised to certain powers.

Let us try to observe the sequence of unit digits, here $U(a^n)$ implies unit digit of a^n

a	$U(a^2)$	$U(a^3)$	$U(a^4)$	$U(a^5)$	$U(a^6)$	$U(a^7)$	$U(a^8)$
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	4	8	6	2	4	8	6
3	9	7	1	3	9	7	1
4	6	4	6	4	6	4	6
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	9	3	1	7	9	3	1
8	4	2	6	8	4	2	6
9	1	9	1	9	1	9	1

Observations from table 1:

1. Unit digits start repeating, after a finite values of n.
2. For the numbers 0, 1, 5 or 6, when each of these digits is raised to any power, the unit digit of the final answer is the number itself.
3. 4 and 9 have a cyclicity of two different digits as their unit digit.
4. 2, 3, 7 and 8 have a cyclicity of four different digits as their unit digit.

Example 1: Find the last digit of the number 9^{197}.

Sol. From table 1, odd power of 9 gives unit digit 9.

Example 2: Find the last digit of the number 2^{50}.

Sol. 2 has a cyclicity of four different digits as their unit digit. $50 = 4 \times 12 + 2$

$U(2^{50}) = U(2^2) = 4$

Example 3: Find the unit digit of the number 8^{971}.

Sol. *8 has a cyclicity of four different digits as their unit digit.* $971 = 4 \times 242 + 3$

$U(8^{971}) = U(8^3) = 2$

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Unit Digit of integers of the form a^n

Let a_0 be the units digit of positive integer a denoted by $U(a) = a_0$. Then the units digit of a^n is determined by the units digit of a_0^n, i.e. $U(a^n) = U(a_0^n)$

Proof.

Let $a = a_m a_{m-1} \ldots a_1 a_0$

$a = (a_m a_{m-1} \ldots a_1)10 + a_0$

$a^n = \left((a_m a_{m-1} \ldots a_1)10 + a_0\right)^n = 10k + a_0^n$

(Using Binomial Theorem)

Clearly, $U(a^n) = U(a_0^n)$.

Example 4: Find the unit digit of 173^{37}.

Sol. $U(173^{37}) = U(3^{37})$

3 has a cyclicity of four digits. $37 = 4 \times 9 + 1$

$U(3^{37}) = U(3^1) = 3$

Example 5: Find the last digit of the number 1989^{1989}.

Sol. To begin let us note that the last digit of the number 1989^{1989} is the same as the last digit of the number 9^{1989}. We write down the last digits of the first few powers of 9: 9, 1, 9, 1, 9,...

Thus, the odd powers of 9 always have their last digit equal to 9. Therefore, the last digit of 1989^{1989} is also 9.

The last two digits of some positive integers of the form a^n

As the n varies, the last two digits of some numbers of the form a^n follow a certain pattern. This pattern is helpful to find the last two digits of these numbers. Let's see some such numbers.

Results:

(i) The last two digits of $5^n \left(n \geq 2\right)$ is 25

(ii) The ordered pair of last two digits of $6^n \left(n \geq 2\right)$ changes with the period "36, 16, 96, 76, 56" as n changes

(iii) The order pair of last to digits of $7^n \left(n \geq 2\right)$ changes with the period "07, 49, 43, 01" as n changes.

(iv) The ordered pair of last two digits of 76^n is always 76.

(v) Last two digits of $(a_1 a_2 \ldots a_{m-1} a_m)^n$ = Last two digits of $(a_{m-1} a_m)^n$

Example 6: Find the last two digits of the number 7^{511}.

Sol. 7 has a cyclicity of four different pairs as their last two digits. $511 = 4 \times 127 + 3$

Last two digits of 7^{511} = Last two digits of $7^3 = 43$

Example 7: Find the last two digits of the numbers 6^{112}.

Sol. 6 has a cyclicity of five different pairs as their last two digits. $112 = 22 \times 5 + 2$

Last two digits of 6^{112} = Last two digits of $6^2 = 36$

Example 8: Find the last two digits of the number 2^{1120}.

Sol. Last two digits of 2^{10} = Last two digits of $1024 = 24$

Last two digits of 2^{20} = Last two digits of $1024 \times 1024 = 76$

$1120 = 20 \times 56$

Last two digits of 2^{1120} = Last two digits of $(2^{20})^{56}$

= Last two digits of $(76)^{56} = 76$

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Exercise – 3

1. Find the last digit of 7^{7777}.
2. What is the units digit of $3^{1999} \times 7^{2000} \times 17^{2001}$?
   (a) 1 (b) 3 (c) 5 (d) 7 (e) 9
3. Find the unit's digit of the expression $25^{6251} + 36^{528} + 73^{54}$
   (a) 4 (b) 0 (c) 6 (d) 5
4. Find the units digit of 3^{100}.
5. The unit's digit in the expression $36^{234}.33^{512}.39^{100} - 54^{29}.25^{123}.31^{512}$ will be
   (a) 8 (b) 0 (c) 6 (d) 5
6. $12^{55}/3^{11} + 8^{40}/16^{10}$ will give which digit at unit's place?
   (a) 4 (b) 6 (c) 8 (d) 0
7. Find the last two digits of the following number
   $101 \times 102 \times 103 \times 197 \times 198 \times 199$
   (a) 54 (b) 74 (c) 64 (d) 84
8. Find last two digits of $6^{100} + 7^{81}$.

1.4 The Euclidean algorithm to find the GCD

Given integers b and c, $c > 0$, we make a repeated application of the division algorithm to obtain a series of equations:

$b = cq + r_1, \quad 0 < r_1 < c$
$c = r_1q_1 + r_2, \quad 0 < r_2 < r_1$
$r_1 = r_2q_2 + r_3, \quad 0 < r_3 < r_2$
⋮ ⋮
$r_{j-3} = r_{j-2}q_{j-2} + r_{j-1}, \quad 0 < r_{j-1} < r_{j-2}$
$r_{j-2} = r_{j-1}q_{j-1} + r_j, \quad 0 < r_j < r_{j-1}$
$r_{j-1} = r_jq_j$

Then the gcd (b, c) of b and c is r_j, the last non zero remainder in the division process.
Moreover, if $(b, c) = bx_0 + cy_0$ then the values of x_0 and y_0 can be obtained by eliminating $r_{j-1}, \ldots, r_2, r_1$ from the above set of equations. i.e. gcd (b, c) of b and c can be written as linear combination of b and c.

Example 1: Find gcd of 4840 and 1512. Also, express the gcd as the linear combination of the given numbers.

Sol. $4840 = 3 \times 1512 + 304$
$1512 = 4 \times 304 + 296$
$304 = 1 \times 296 + 8$
$296 = 37 \times 8$
$304 - 296 = 8$
$304 - (1512 - 4 \times 304) = 8$
$5 \times 304 - 1512 = 8$
$5 \times 4840 - 16 \times 1512 = 8$

Example 2: Express the gcd of (63, 48) as the linear combination of 126 and 48.

Sol. $63 = 48 + 15$

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$48 = 15 \times 3 + 3$

$15 = 3 \times 5$

$3 = 48 - 15 \times 3$

$= 48 - (63 - 48) \times 3$

$3 = 4 \times 48 - 3 \times 63$

Example 3: By using Euclidean algorithm find the gcd of (i) 10703 and 2842 (ii) 3645 and 2357. Also express the gcd as the linear combination of the given numbers.

Sol. $10703 = 3 \times 2842 + 2177$

$2842 = 1 \times 2177 + 665$

$2177 = 3 \times 665 + 182$

$665 = 3 \times 182 + 119$

$182 = 1 \times 119 + 63$

$119 = 1 \times 63 + 56$

$63 = 1 \times 56 + 7$

$56 = 8 \times 7 + 0$

H.C.F = 7

$63 - 56 = 7$

$63 - (119 - 63) = 7$

$2 \times 63 - 119 = 7$

$2 \times (182 - 119) - 119 = 7$

$2 \times 182 - 3 \times 119 = 7$

$11 \times 182 - 3 \times 665 = 7$

$11 \times 2177 - 36 \times 665 = 7$

$47 \times 2177 - 36 \times 2842 = 7$

$47 \times 10703 - 177 \times 2842 = 7$

(ii) $3645 = 1 \times 2357 + 1288$

$2357 = 1 \times 1288 + 1069$

$1288 = 1 \times 1069 + 219$

$1069 = 4 \times 219 + 193$

$219 = 1 \times 193 + 26$

$193 = 7 \times 26 + 11$

$26 = 2 \times 11 + 4$

$11 = 2 \times 4 + 3$

$4 = 3 \times 1 + 1$

$4 - 3 = 1$

$4 - (11 - 2 \times 4) = 1$

$3 \times 4 - 11 = 1$

$3 \times 26 - 7 \times 11 = 1$

$52 \times 26 - 7 \times 193 = 1$

$52 \times 219 - 59 \times 193 = 1$

$288 \times 219 - 59 \times 1069 = 1$

$288 \times 1288 - 347 \times 1069 = 1$

$635 \times 1288 - 347 \times 2357 = 1$

$635 \times 3645 - 982 \times 2357 = 1$

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Properties: Let a, b and n are integers

(i) If gcd(a, b) = 1, then gcd(a, na + b) = gcd(b, a + nb) = 1
(ii) gcd(a, b) = gcd(a, a + nb) = gcd(an + b, b)
Corollary: gcd(a, b) = gcd(a, a – b)

Proof.

(i) We prove this by contradiction
let gcd(a, na + b) = g > 1
$\Rightarrow$ a = gp and na + b = gq, where gcd(p, q) = 1
b = gq - ngp = g(q - np)
$\Rightarrow$ gcd(a, b) $\geq$ g > 1 which contradicts our assumption.

(ii) let gcd(a, b) = g > 1
$\Rightarrow$ a = gp and b = gq, here p and q are co-prime
gcd(a, an + b) = gcd(gp , ngp + gq)
= g × gcd(p, np + q)
= g × 1 = g from (i)

Example 4: Find the GCD of the numbers 2n + 13 and n + 7 for any natural number n.

Sol. We have gcd(2n + 13, n + 7)
= gcd(2n + 13 – n – 7, n + 7) = gcd (n + 7, n + 6) = gcd (1, n + 6) = 1

Example 5: Prove that the fraction $\frac{12n+1}{30n+2}$ cannot be reduced for any natural number n.

Sol. g.c.d (30n + 2, 12n + 1) = g.c.d (30n + 2 – 2(12n + 1), 12n + 1)
= g.c.d (6n, 12n + 1) = g.c.d (6n, 12n + 1 – 2(6n))
= g.c.d (6n, 1) = 1

Example 6: Find gcd $(2^{100} - 1, 2^{120} - 1)$.

Sol. gcd $(2^{100} - 1, 2^{120} - 1)$ = gcd $(2^{100} - 1, 2^{120} - 2^{100})$
= gcd $(2^{100} - 1, 2^{100}(2^{20} - 1))$
= gcd $(2^{100} - 1, 2^{20} - 1)$
(Since $2^{100} - 1$ and 2^{100} are coprime)
By repeating these steps, we get
= g.c.d $(2^{20} - 1, 2^{20} - 1) = 2^{20} - 1$

(M2) $2^{120} - 1 = (2^{20})^6 - 1 \Rightarrow$ One factor $2^{20} - 1$

Because $x^6 - 1 = (x - 1)(x^5 + x^4 + x^3 + x^2 + x + 1)$

$2^{100} - 1 = (2^{20})^5 - 1 \Rightarrow$ one factor $2^{20} - 1$

$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$

g.c.d $\left((x^5 + x^4 + x^3 + x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)\right)$

g.c.d $\left((x^5 + x^4 + x^3 + x^2 + x + 1) - x(x^4 + x^3 + x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)\right)$

g.c.d $(1 \cdot x^4 + x^3 + x^2 + x + 1) = 1$

$\Rightarrow$ gcd $(2^{120} - 1, 2^{100} - 1) = 2^{20} - 1$

Example 7: Find gcd (111 ... 11, 11 ... 11), where there are one hundred 1's in the decimal representation of the first number and sixty 1's in the decimal representation of the second number.

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Sol. $100\ 1's = 111.....1 = \frac{1}{9}(10^{100} - 1)$

$60\ 1's = 111.....1 = \frac{1}{9}(10^{60} - 1)$

$$g.c.d\left(\frac{10^{100}-1}{9}, \frac{10^{60}-1}{9}\right) = g.c.d\left(\frac{10^{60}-1}{9}, \frac{10^{60}(10^{40}-1)}{9}\right)$$

$$= g.c.d\left(\frac{10^{60}-1}{9}, \frac{10^{40}-1}{9}\right) = g.c.d\left(\frac{10^{40}(10^{20}-1)}{9}, \frac{10^{40}-1}{9}\right)$$

As gcd of 10^{40} and $10^{40} - 1$ is 1.

$$= \frac{10^{20}-1}{9} = (111....1)_{20\ 1's}$$

Example 8: The gcd of two positive integers is 81 and their lcm is 5103. Find the numbers.

Sol. Let numbers be 81a & 81b, here a & b are co-prime

L.C.M = 81ab = 5103

$ab = 63 = 3^2 \times 7^1$

$a = 3^2$ & b = 7 or a = 63 & b = 1

Numbers are (729, 567) or (5103, 81)

Exercise – 4

1. The greatest number that divides 22176 and 22396 without leaving any remainder is
   (a) 48 (b) 42 (c) 38 (d) 52 (e) None of these
2. Find all integers x and y such that hcf (x, y) = 8 and lcm (x, y) = 64.
3. The product of two positive integers is 9984 and the greatest common factor of those integers equals the difference between them. What are the two integers?
4. Find the gcd (2n + 1, 5n + 4).
5. Find the gcd (2020×2021, 2019×2022)?
6. Find the number of pairs m > n of positive integers such that 1 = gcd(n + 1, m + 1) = gcd (n + 2, m + 2) =....gcd(m, 2m – n) and m + n < 20.
7. Find the GCD of the numbers $5^3 + 7^3$ and $5^5 + 7^5$.
8. [AMC 10, 2018] How many ordered pairs (a, b) of positive integers satisfy the equation a.b + 63 =20lcm(a, b) + 12gcd(a, b), where gcd(a, b) denotes the greatest common divisor of a and b, and lcm(a, b) denotes their least common multiple?
9. Three numbers which are coprime to each other are such that the product of first two is 551 and that of the last two is 1073. Find the sum of the three numbers.
10. [USAMO 1972] The symbols (a, b,...., g) and [a, b,....., g] denote the greatest common divisor and least common multiple, respectively, of the positive integers a, b,...., g. For example (3, 16, 18) = 3 and [6, 15] = 30.

    Prove that $\frac{[a,b,c]^2}{[a,b][b,c][c,a]} = \frac{(a,b,c)^2}{(a,b)(b,c)(c,a)}$.

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1.5 Congruent Modulo

Modulo:

This operation gives the remainder when one number is divided by the other. If we have two integers **a** and **b**, then **a modulo b**, denoted by **a mod b** gives the remainder when **a** is divided by **b**.

Congruent Modulo:

Two integers a and b are said to be **congruent modulo** m, denoted by $a \equiv b \pmod{m}$ if a and b have the same remainder when they are divided by a non-zero integer m. If the remainders are different, then a and b are said to be not congruent modulo m, denoted $a \not\equiv b \pmod{m}$

By the definition of congruence, the following four equivalent relations are obvious:

$a \equiv b \pmod{m} \Leftrightarrow a - b = km \Leftrightarrow a - b \equiv 0 \pmod{m} \Leftrightarrow m \mid (a - b)$

Basis properties of Congruent Modulo

(I) If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.

(II) If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$(a+c) \equiv (b+d) \pmod{m}$, $(a-c) \equiv (b-d) \pmod{m}$

(III) If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$

(IV) If $a \equiv b \pmod{m}$ then $a^n \equiv b^n \pmod{m}$ for all natural numbers n

(V) If $ac \equiv bc \pmod{m}$ and $(c, m) = 1$, then $a \equiv b \pmod{m}$

Proofs.

(I) If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

$a - b = km$, $b - c = nm$, here k and n are integers.

$a - b + b - c = m(k + n)$

$a - c = m(k + n)$

$a \equiv c \pmod{m}$

(II) $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$

$a - b = km$, $c - d = nm$, here k and n are integers.

Adding these two, we get

$a + c - b - d = m(k + n)$

$(a+c) \equiv (b+d) \pmod{m}$

Similarly, we can prove $(a-c) \equiv (b-d) \pmod{m}$

(III) $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$

$a = km + b$, $c = nm + d$, here k and n are integers.

Multiplying these two, we get

$ac = knm^2 + dkm + bnm + bd$

$ac - bd = m(knm + dk + bn)$

$\Rightarrow ac \equiv bd \pmod{m}$.

(IV) If $a \equiv b \pmod{m}$

From (iii), $aa...aa \equiv bb...bb \pmod{m}$

$\Rightarrow a^n \equiv b^n \pmod{m}$

(V) $ac \equiv bc \pmod{m}$ and $(c, m) = 1$

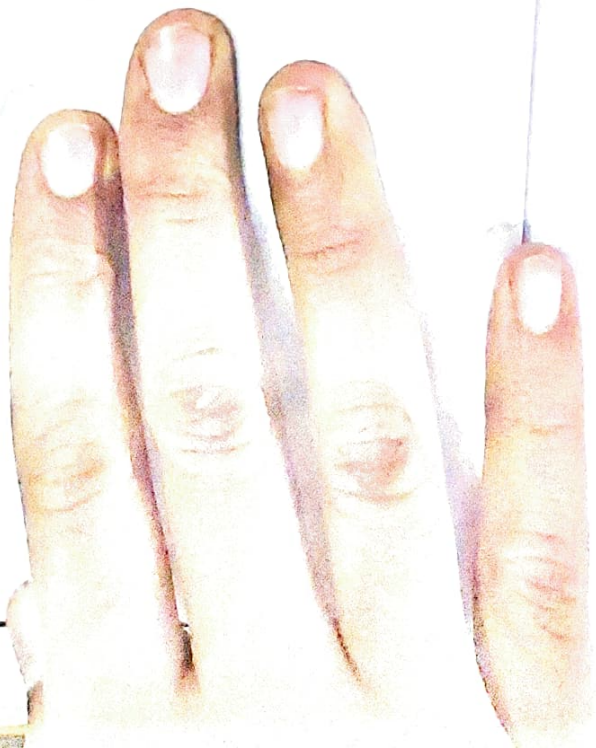
$ac - bc = km$, here k is an integer.

$c(a - b) = km$

$\Rightarrow$ m divides $c(a - b)$

$\Rightarrow$ m divides $a - b$

$\Rightarrow a \equiv b \pmod{m}$



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Example 1: Reduce 6^{100} modulo 7

Sol. Since $6 - (-1) = 6 + 1 = 7$,

we can say that $6 \equiv -1 \pmod 7$

Raising this congruence to the power of 100

we have $6^{100} \equiv (-1)^{100} \pmod 7$

that is $6^{100} \equiv 1 \pmod 7$

Example 2: Find the remainder when 3^{99} is divided by 13.

Sol. Since $27 - 1 = 26$,

we can say that $27 \equiv 1 \pmod{13}$

$3^3 \equiv 1 \pmod{13}$

Raising this congruence to the power of 33 we have $3^{99} \equiv (1)^{33} \pmod{13}$

that is $3^{99} \equiv 1 \pmod{13}$

Example 3: Find the remainder when 5^{20} is divided by 7.

Sol. Since $125 - (-1) = 126$,

we can say that $125 \equiv -1 \pmod 7$.

$5^3 \equiv -1 \pmod 7$

Raising this congruence to the power of 6 we have $5^{18} \equiv (-1)^6 \pmod 7$

that is $5^{18} \equiv 1 \pmod 7$ .....(1)

Also $5^2 \equiv 4 \pmod 7$ .....(2)

Multiply 1 and 2 we get

$5^{20} \equiv 4 \pmod 7$

Example 4: Find the remainder of 3^{2000} when it is divided by 13.

Sol. $3^3 = 27 \equiv 1 \pmod{13}$ provides the method for reducing the power of 3, it follows that

$3^{2000} \equiv \left(3^3\right)^{666} . 3^2 \equiv 3^2 \equiv 9 \pmod{13}$

Thus, the remainder is 9.

Note: For finding the remainder of a large power of a positive integer, it is important to find the minimum power which gives remainder 1 or see if the remainders are constant as the power changes.

Example 5: Find the remainder when $2^{11} + 3^{11}$ is divided by 5.

Sol. We know that $a^n + b^n$ is divisible by $(a + b)$ if n is odd.

Here, $a = 2$, $b = 3$ and $n = 11$ (i.e. odd)

So, remainder is 0.

Example 6: Find the remainder when $11^{50} . 17^{12}$ is divided by 6.

Sol. Since $11 - (-1) = 11 + 1 = 12$,

we can say that $11 \equiv -1 \pmod 6$

Raising this congruence to the power of 50

we have $11^{50} \equiv (-1)^{50} \pmod 6$

that is $11^{50} \equiv 1 \pmod 6$ .....(1)

Since $17 - (-1) = 17 + 1 = 18$,

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we can say that: $17 \equiv -1 \pmod 6$
Raising this congruence to the power of 12
we have $17^{12} \equiv (-1)^{12} \pmod 6$
that is $17^{12} \equiv 1 \pmod 6$.....(2)
multiply (1) and (2)
$11^{50}.17^{12} \equiv 1 \pmod 6$

Example 7: Prove that $30^{99} + 61^{100}$ is divisible by 31.

Sol. Since $30 - (-1) \equiv 30 + 1 \equiv 31$,
we can say that $30 \equiv -1 \pmod{31}$
Raising this congruence to the power of 99
we have $30^{99} \equiv (-1)^{99} \pmod{31}$
that is $30^{99} \equiv -1 \pmod{31}$.....(1)
Since $61 + 1 \equiv 62$,
we can say that: $61 \equiv -1 \pmod{31}$
Raising this congruence to the power of 100
we have $61^{100} \equiv (-1)^{100} \pmod{31}$
that is $61^{100} \equiv 1 \pmod{31}$.....(2)
Add (1) and (2)
$30^{99} + 61^{100} \equiv 0 \pmod{31}$

Example 8: Prove that

(a) $43^{101} + 23^{101}$ is divisible by 66
(b) $a^n - b^n$ is divisible by $a - b$

Sol. (a) $43 \equiv -23 \pmod{66}$
$43^{101} \equiv (-23)^{101} \pmod{66}$
Add 23^{101} to both sides
$43^{101} + 23^{101} \equiv (-23)^{101} + 23^{101} \pmod{66}$
$43^{101} + 23^{101} \equiv 0 \pmod{66}$
(b) $a \equiv b \pmod{a - b}$
$a^n \equiv b^n \pmod{a - b}$
$a^n - b^n \equiv 0 \pmod{a - b}$

Example 9: Find the smallest positive integer k such that $2^{69} + k$ is divisible by 127.

Sol. $2^7 \equiv 1 \pmod{127}$ implies $2^{7m} \equiv \left(2^7\right)^m \equiv 1^m \equiv 1 \pmod{127}$, hence
$(2^7)^9(2^6) \equiv 2^6 \pmod{127}$
$2^{69} \equiv 64 \pmod{127}$
Therefore, the minimum value of k is equal to $127 - 64 = 63$.

Example 10: Prove that $1^n + 2^n + ... + \left(n-1\right)^n$ is divisible by n for odd n.

Sol. $n - r \equiv -r \pmod n$
$(n - r)^n \equiv (-r)^n \pmod n$
$(n - r)^n + r^n \equiv 0 \pmod n$
$\Rightarrow 1^n + (n - 1)^n \equiv 0 \pmod n$
$2^n + (n - 2)^n \equiv 0 \pmod n$

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$$\left(\frac{n-1}{2}\right)^n + \left(\frac{n+1}{2}\right)^n \equiv 0 \mod(n)$$

Add these we get

$1^n + 2^n + \ldots + (n-1)^n \equiv 0 \pmod n$

(M2) $a^n + b^n$ divisible by $(a + b)$ if n is odd

$$1^n + 2^n + \ldots + \left(\frac{n-1}{2}\right)^n + \left(\frac{n+1}{2}\right)^n + \ldots + (n-2)^n + (n-1)^n$$

$\Rightarrow 1^n + (n-1)^n$ is divisible by n

$\Rightarrow 2^n + (n-2)^n$ is divisible by n

$\left(\frac{n-1}{2}\right)^n + \left(\frac{n+1}{2}\right)^n$ is divisible by n

$\Rightarrow$ Their sum is divisible by n

Example 11: What is the remainder when $6^{273} + 8^{273}$ is divided by 49?

Sol. In general, for odd positive integer n,

$$a^n + b^n = (a+b)\left(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots + b^{n-1}\right)$$

So that

$$6^{273} + 8^{273} = (6+8)\left(6^{272} - 6^{271}.8 + 6^{270}.8^2 - \ldots + 8^{272}\right) = 14M,$$

where $M = 6^{272} - 6^{271}.8 + 6^{270}.8^2 - \ldots + 8^{272}$. Furthermore,

$$M \equiv \underbrace{(-1)^{272} - (-1)^{271} + (-1)^{270} - \ldots + 1}_{273 \text{ terms}} \equiv 273 \equiv 0 \pmod 7$$

Therefore 7 | M, hence 49 | 14 M, i.e. the remainder is 0.

Example 12: Find the remainder of the number $2005^{2007^{2009}}$ when divided by 7.

Sol. First of all $2005^{2007^{2009}} \equiv 3^{2007^{2009}} \pmod 7$.

Since $3^3 \equiv -1 \pmod 7$ yields

$3^6 \equiv \left(3^3\right)^2 \equiv 1 \pmod 7$

$2007^{2009} \equiv 3^{2009} \equiv 3 \pmod 6$

It follows that $2007^{2009} = 6k + 3$ for some positive integer k. Therefore

$2005^{2007^{2009}} \equiv 3^{6k+3} \equiv 3^3 \equiv 6 \pmod 7$

Thus, the remainder is 6.

Example 13: Find the remainder when $13^{73} + 14^3$ is divided by 11.

Sol. $13 \equiv 2 \pmod{11}$ and $14 \equiv 3 \pmod{11}$.

Hence, $143 \equiv 33 \equiv 5 \pmod{11}$

Also $25 \equiv -1 \pmod{11}$. Hence $270 \equiv 1 \pmod{11}$ and $273 \equiv 8 \pmod{11}$

$13^{73} \equiv 2^{73} \equiv 8 \pmod{11}$

$\therefore 13^{73} + 14^3 \equiv 8 + 5 \equiv 2 \pmod{11}$

Example 14: Find smallest positive integer n such that $1000 \le n \le 1100$ and $1111^n + 1222^n + 1333^n + 1444^n$ is divisible by 10.

Sol. Let $N = 1111^n + 1222^n + 1333^n + 1444^n$. Then $N \equiv 1^n + 2^n + 3^n + 4^n \pmod{10}$

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Unit digit of $1^n + 2^n + 3^n + 4^n$ must be zero for divisibility by 10.

For n = 1, 2, 3; unit digit is zero

At n = 4; unit digit is non-zero

Similarly at n = 4k+1, 4k+2,4k+3; unit digit is zero.

Therefore n = 1001

Example 15: Prove that for any odd natural number n, the number $1^{2007} + 2^{2007} + ... + n^{2007}$ is not divisible by n + 2.

Sol. By taking modulo n + 2, and partition the terms as groups of two each.

$$1^{2007} + 2^{2007} + .. + n^{2007}$$

$$= 1 + \left[2^{2007} + n^{2007}\right] + ... + \left[\left(\frac{n+1}{2}\right)^{2007} + \left(\frac{n+3}{2}\right)^{2007}\right]$$

$$\equiv 1 + \left[2^{2007} + (-2)^{2007}\right] + ... + \left[\left(\frac{n+1}{2}\right)^{2007} + \left(-\frac{n+1}{2}\right)^{2007}\right] \bmod(n+2)$$

$$\equiv 1 \pmod{n+2}$$

Thus, the conclusion is proven.

Example 16: Show that the square of any integer is of the form 4k or 8k + 1.

Sol. $n \equiv 0, \pm1, \pm2, \pm3, 4 \pmod 8$

$n^2 \equiv 0, 1, 4, 9, 16 \pmod 8$

$n^2 \equiv 0, 1, 4 \pmod 8$

$\Rightarrow n^2 = 8k, 8k+1, 8k+4$

$\Rightarrow n^2 = 4k, 8k+1$

Example 17: Show that the square of any integer is of the form 3k or 3k + 1.

Sol. $n \equiv 0, \pm1 \pmod 3$

$n^2 \equiv 0, 1 \pmod 3$

$\Rightarrow n^2 = 3k, 3k+1$

Example 18: Show that the cube of any integer is of the form 9k or 9k +1 or 9k + 8.

Sol. $n \equiv 0, \pm1, \pm2, \pm3, \pm4 \pmod 9$

$n^3 \equiv 0, \pm1, \pm8, \pm27, \pm64 \pmod 9$

$n^3 \equiv 0, \pm1 \pmod 9$

$\Rightarrow n^2 = 9k, 9k+1, 9k+8$

Example 19: Let f(x) denotes a polynomial with integral coefficients. If $a \equiv b \pmod m$, then prove that $f(a) \equiv f(b) \pmod m$

Sol. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 + a_0$

$a \equiv b \pmod m$

$a^r \equiv b^r \pmod m$

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$\Rightarrow a_r a^r \equiv a_r b^r \pmod m$

$\Rightarrow a_n a^n + a_{n-1}a^{n-1} + \ldots + a_0 \equiv a_n b^n + a_{n-1}b^{n-1} + \ldots + a_0 \pmod m$

$f(a) \equiv f(b) \pmod m$

Example 20: Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Sol. $2222^{5555} + 5555^{2222} \equiv (3^{5555} + 4^{2222}) \pmod 7$

$\equiv 243^{1111} + 16^{1111} \pmod 7$

$\equiv (-2)^{1111} + 2^{1111} \pmod 7$

$\equiv 0 \pmod 7$

$2222^{5555} + 5555^{2222} \equiv 0 \pmod 7$

Example 21: Prove that any natural number is congruent to the sum of its digits modulo (a) 3; (b) 9

Sol. Consider the number

$\overline{a_1 a_2 \ldots a_n} = a_1 10^{n-1} + a_2 10^{n-2} + a_{n-1} 10^1 + a_n$

Clearly, $10 \equiv 1 \pmod 9$. Thus, $10^k \equiv 1 \pmod 9$ for any natural k. So, we have

$a_1 10^{n-1} + a_2 10^{n-2} + \ldots + a_{n-1} 10^1 + a_n \equiv a_1 + a_2 + \ldots a_n \pmod 9$

Hence, we can deduce that a number is divisible by 3 or 9 if the sum of digits of n is divisible by 3 or 9 respectively

Note: From the above example, a natural number is congruent to its sum of digits modulo 9. Let us have a number N, having sum of digits as S_1 and sum of digits of S_1 as S_2 and so on, till we get a single digit say S_k. If N is divisible by 9, then $S_k = 9$. If N is not divisible by 9 then S_k is equal to the remainder when the number N is divided by 9.

Proof:

$N \equiv S_1 \equiv S_2 \equiv \ldots \equiv S_k \pmod 9$

$\Rightarrow S_k$ is congruent to N mod 9 means N and S_k both are giving same remainder when divided by 9.

Case I: If N is divisible by 9, so S_k also divisible by 9, therefore $S_k = 0$ or 9, but S_k cannot be zero, as the sum f digits always be a positive integer. Therefore $S_k = 9$

Case II: If N is not divisible by 9, so S_k also not divisible by 9. As S_k is a single digit number, when divided by 9, the remainder is S_k itself. Therefore, S_k is equal to the remainder when N divided by 9.

Example 22: The sum of the digits was calculated for number 2^{100}, then the sum of the digits was calculated for the resulting number and so on, until a single digit is left. Which digit is this?

Sol. $2^3 \equiv -1 \pmod 9$

$2^{100} \equiv -2 \equiv 7 \pmod 9$

Let $2^{100} = a_1 a_2 a_3 \ldots a_n$

$2^{100} \equiv a_1 + a_2 + a_3 \ldots + a_n \equiv 7 \pmod 9$

if $a_1 + a_2 + a_3 \ldots + a_n = b_1 b_2 b_3 \ldots b_m$

$b_1 b_2 b_3 \ldots b_m \equiv b_1 + b_2 + b_3 \ldots + b_m \equiv 7 \pmod 9$

$\Rightarrow$ Sum of digits at every step is 7 modulo 9.

$\Rightarrow$ Final number is = 7

Example 23: Prove that $\overline{a_1 a_2 \ldots a_n} = a_n - a_{n-1} + \ldots + (-1)^{n-1} a_1 \pmod{11}$

Sol. $10 \equiv -1 \pmod{11}$, so even powers of 10 are congruent to 1, and odd power to -1, modulo 11.

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$a_1a_2...a_n = 10^{n-1} \cdot a_1 + 10^{n-2} \cdot a_2 + ... + a_n$

$\equiv a_n - a_{n-1} + ... + (-1)^{n-1} a_1 \pmod{11}$

Example 24: The sum of the digits of a three-digit number equals 7. Prove that this number is divisible by 7 if and only if its two last digits are equal.

Sol. Since $2(a + b + c) \equiv 0 \pmod 7$,
we get $\overline{abc} = 100a + 10b + c \equiv 2a + 3b + c \pmod 7 \equiv b - c \pmod 7$.
Therefore, abc is divisible by 7 if and only if $b - c$ is divisible by 7. Taking into account that both digits b and c are less than 7, the only way this difference can be divisible by 7 is if the two digits are equal.

Exercise – 5

1. Prove that for any natural number n the expression $A = 2903^n - 803^n - 464^n + 261^n$ is divisible by 1897.
2. Show that the square of an odd integer is $\equiv 1 \pmod 8$.
3. Show that the square of an integer is $\equiv 0$ or $1 \pmod 3$.
4. Show that the square of an integer is $\equiv 0, 1, -1 \pmod 5$.
5. If an integer n is co-prime to 6, then show that $n^2 \equiv 1 \pmod{24}$.
6. Prove that $n^2 + 1$ is not divisible by 3 for any integer n.
7. Find the remainder when the number $10^{10} + 10^{100} + 10^{1000} + ... + 10^{1000000000}$ is divided by 7.
8. What is the remainder of 123456789^4 when it is divided by 8?
9. What is the remainder when $9^{9^{9^9}}$ is divided by 11?
10. (i) Find the remainder when 2^{50} and 41^{65} are divided by 7.
    (ii) Find the remainder when $13^{71} + 14^3$ is divided by 11.
11. (a) Find all positive integers n for which $2^n - 1$ is divisible by 7.
    (b) Prove that there is no positive integer n for which $2^n + 1$ is divisible by 7.
12. Find the last two digits of 2^{999}.
13. The six-digit number $\overline{abcdef}$ satisfies the property that $\overline{def} - \overline{abc}$ is divisible 7. Prove that the number itself is divisible by 7. Using this information attempt the following.
    (a) State and prove a test for divisibility by 7.
    (b) State and prove a test for divisibility by 13.
14. Is it possible for the sum of the first several natural numbers to be 1989?
15. Prove that if you reverse the order of the digits in any natural number and subtract the result from the initial number, then the difference is divisible by 9.
16. Prove that a power of 2 cannot end with four equal digits.

1.6 Divisibility Tests

Divisibility Tests for 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

- 2, 5, 10: Use that any number is always congruent to its unit digit modulo 2,5 or 10
- 4: Use that any number is always congruent to its last two digits modulo 4
- 8: Use that any number is always congruent to its last three digits modulo 8
- 3, 9: Use that any number is always congruent to sum of its digits modulo 3 or 9
- 11: If the difference between the sum of the odd-numbered digits and the sum of the even-numbered digits, counted from right to left, is divisible by 11, then the number is divisible by 11.

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Reason: Any number is always congruent to a number (formed by addition and subtraction of alternate digits of the given number) modulo 11, as discussed in example 24 of 1.5.

- 7: Let N= ab be a number. (where b is it unit digit and a represents the remaining digits). Then N is divisible by 7 iff a – 2b is divisible by 7.
  Reason: $10a + b \equiv 10(a - 2b) \pmod 7$

Example 1: Find the least natural number divisible by 36 which has all 10 digits in its decimal representation

Sol. Sum of digits = 1 + 2 +...+ 9 = 45 which is divisible by 9. For divisibility by 4, last two digits must be divisible by 4. So, we place the largest two digit multiple of 4 (i.e. 96) at the last, and arrange the rest in the increasing order. Hence, the least number is 1023457896.

Example 2: Can a number written with one hundred 0's, one hundred 1's, and one hundred 2's be a perfect square?

Sol. No. The sum of the digits of any number such as described in the problem is 100(0 +1 + 2) =300, which is divisible by 3 but not by 9.

Example 3: The sum of the digits was calculated for $13^{2222} - 2^{4444}$ then, the sum of the digits was calculated for the resulting number and so on, until a single digit is left. Which digit is this?

Sol. $13^{2222} - 2^{4444} = 169^{1111} - 16^{1111}$ is divisible by 169 – 16 = 153 which is a multiple of 9.
So, sum of digits is multiple of 9.
So, the number obtained from sum is divisible by 9.
Therefore, sum of its digits is also multiple of 9.
It means every sum is divisible by 9. Therefore, at end we get 9.

Example 4: Prove that the number 111...11(2n ones) is composite.

Sol. Difference between the sum of the odd-numbered digits and the sum of the even-numbered digits is zero. Therefore, number is divisible by 11.

Example 5: Is it possible to form two numbers using only the digits 2, 3, 4, 9 such that one of them is 19 times greater than the other?

Sol. The answer is no. Indeed, if you multiply any of the given four digits by 9, then you cannot obtain one of these digits again in the units place of the answer.

Exercise – 6

1. Find the last two digits of the following number 75 × 35 × 47 × 63 × 71 × 87 × 82
   (a)50 (b) 70 (c) 30 (d) 90
2. Ram multiplied two two-digit numbers on the blackboard. Then he changed all the digits to letters (different digits were changed to different letters, and equal digits were changed to the same letter). He obtained AB×CD = EEFF. Prove that Ram made a mistake somewhere.
3. Find at least one 100-digit number without zeroes in its decimal representation, which is divisible by the sum of its digits.
4. How many four-digit numbers with the two middle digits 97 are divisible by 45?
5. Find the smallest number written only with ones which is divisible by 333...33 (one hundred 3's in the representation).
6. Prove that the number $\overline{a_1 a_2 \ldots a_0 a_0 \ldots a_2 a_1}$ is composite.

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1.7 Perfect Square Numbers

A perfect square is a number that can be expressed as the product of an integer by itself or as the second exponent of an integer. For example, 25 is a perfect square.

Basic Properties of Perfect Square Numbers

(i) The units digit of a perfect square can be 0, 1, 4, 5, 6 and 9 only. It suffices to check the property for $0^2, 1^2, 2^2, \ldots, 9^2$.

(ii) If the prime factorization of a natural number n is $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ then n is a perfect square $\Leftrightarrow$ each α_i is even $\Leftrightarrow r(n)$ is odd; where r(n) denotes the number of positive divisors of n.

(iii) For any perfect square number *n*, the number of its tail zeros (i.e. the digit 0s on its right end) must be even, since in the prime factorization of *n* the number of factor 2 and that of factor 5 are both even.

(iv) $n^2 \equiv 1$ or 0 modulo 2,3,4

It suffices to check the numbers of the forms $(2m)^2$ and $(2m+1)^2$ by taking modulo 2 and modulo 4 respectively; the numbers of the forms of $(3m)^2$ and $(3m \pm 1)^2$ by taking modulo 3.

(v) $n^2 \equiv 0,1$ or $4 \pmod 8$

It suffices to check the conclusion for $(4m \pm 1)^2, (4m)^2, (4m+2)^2$, where m is any integer.

(vi) An odd perfect square number must have an even tens digit (If one-digit perfect squares 1^2 and 3^2 are considered as 01 and 09 respectively)

It is easy to see the reasons: For $n > 3^2, n^2 = (10a+b)^2 = 100a^2 + 20ab + b^2$. The number $100a^2 + 20ab$ has units digit 0 and an even tens digit. If b is an odd digit, then the tens digit carried from b^2 must be even, so the tens digit of n^2 must be even.

(vii) If the tens digit of n^2 is odd, then the units digit of n^2 must be 6. Continue the analysis in (vi). If the tens digit carried from b^2 is odd, then b = 4 or 6 only, so $b^2 = 16$ or 36, i.e. the unit digit of n^2 must be 6.

Example 1: Prove that for any integer k, all the numbers of the forms $3k+2$; $4k+2$; $4k+3$; $5k+2$; $5k+3$; $8k \pm 2$; $8k \pm 3$; $8k+7$ cannot be perfect squares.

Sol. $3k+2 \equiv 2 \pmod 3$, $4k+2 \equiv 2 \pmod 4$, $4k+3 \equiv 3 \pmod 4$ implies they cannot be perfect squares. $5k+2$ has units digit 2 or 7 and $5k+3$ has units digit 3 or 8, so they cannot be perfect squares also. All the numbers $8k \pm 2$; $8k \pm 3$; $8k+7$ have no remainders 0, 1 or 4 modulo 8, so they cannot be perfect squares.

Example 2: Prove that any positive integer $n \geq 10$ cannot be a perfect square number if it is formed by the same digits.

Sol. If the used digit is odd, then the conclusion is proven by the property (vi). If the used digit is 2 or 8, the conclusion is obtained at once from the property (i). If the used digit is 6, the conclusion is obtained by the property (vii). Finally, if the used digit is 4, let $n^2 = \underbrace{44\ldots44}_{k}$ with $k \geq 2$, then n = 2m, i.e. $n^2 = 4m^2$, so $m^2 = \underbrace{11\ldots11}_{k}$ a contradiction from above discussion. Thus, the conclusion is proven for all the possible cases.

Example 3: Prove that the sum of 1 and the product of any four consecutive integers must be a perfect square, but the sum of any five consecutive perfect squares must not be a perfect square.

Sol. Let a, a + 1, a + 2, a + 3 are four consecutive integers. Then

$a(a+1)(a+2)(a+3) + 1 = [a(a+3)][(a+1)(a+2)] + 1$

$= (a^2 + 3a)(a^2 + 3a + 2) + 1 = [(a^2 + 3a + 1) - 1][(a^2 + 3a + 1) + 1] + 1$

$= (a^2 + 3a + 1)^2$

Hence Proved.

Now let $(n-2)^2, (n-1)^2, n^2, (n+1)^2, (n+2)^2$ be any five consecutive perfect squares.

$(n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 = 5n^2 + 10 = 5(n^2 + 2)$

If $5(n^2 + 2)$ is a perfect square, then 5 divides $n^2 + 2$.

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∴ Unit digit of n^2 is 3 or 7 but this is impossible.
Thus, the second conclusion is also proven.

Example 4: The square of an integer is called a perfect square number. If x is a perfect square number, then its next square is

(a) $x + 1$ (b) $x^2 + 1$
(c) $x^2 + 2x + 1$ (d) $x + 2\sqrt{x} + 1$

Sol. Since $x \geq 0$, so $x = \left(\sqrt{x}\right)^2$, and its next perfect square is $\left(\sqrt{x}+1\right)^2 = x + 2\sqrt{x} + 1$, the answer is (D).

Example 5: Given that five-digit number $\overline{2x9y1}$ is a perfect square number. Find the value of $3x + 7y$.

Sol. We use estimation method to determine x and y. Let $A^2 = \overline{2x9y1}$. Since $141^2 = 19881 < A^2$ and $175^2 = 30625 > A^2$ so $141^2 < A^2 < 175^2$ the units digit of A^2 is 1 implies that the units digit of A is 1 or 9 only. Therefore, it is sufficient to check 151^2, 161^2, 171^2, 159^2, 169^2 only, so we find that $161^2 = 25921$ satisfies all the requirements, and other numbers cannot satisfy the all requirements Thus, $x = 5, y = 2$, so that $3x + 7y = 15 + 14 = 29$.

Example 6: Find the number of the pairs (x, y) of two positive integers, such that $N = 23x + 92y$ is a perfect square number less than or equal to 2392.

Sol. $N = 23x + 92y = 23(x + 4y)$ and 23 is a prime number implies that $x + 4y = 23m^2$ for some positive integer m, so

$$N = 23^2 m^2 \leq 2392 \Rightarrow m^2 \leq \frac{2392}{23^2} = \frac{104}{23} < 5$$

Hence $m^2 = 1$ or 4, i.e. $m = 1$ or 2

When $m^2 = 1$, then $x + 4y = 23$ or $x = 23 - 4y$. Since x, y are two positive integers, so $y = 1, 2, 3, 4, 5$ and $x = 19, 15, 11, 7, 3$ correspondingly. When $m^2 = 4$, then $x + 4y = 92$ or $x = 92 - 4y$, so y can take each positive integer value from 1 through 22, and x then can take the corresponding positive integer values given by $x = 92 - 4y$. Thus, the number of qualified pairs (x, y) is $5 + 22$, i.e. 27.

Example 7: Prove that 2006 cannot be expressed as the sum of ten odd perfect square numbers.

Sol. We prove by contradiction. Suppose that 2006 can be expressed as the sum of ten odd perfect square numbers i.e.

$$2006 = x_1^2 + x_2^2 + ... + x_{10}^2$$

Where $x_1, x_2 ....., x_{10}$ are all odd numbers. When taking modulo 8 to both sides, the left-hand side is 6, but the right-hand side is 2, a contradiction. Thus, the assumption is wrong, and the conclusion is proven.

Example 8: Find all the natural number n, such that $n^2 - 19n + 91$ is perfect square.

Sol. (i) When $n > 10$, then $n - 9 > 0$, so $n^2 - 19n + 91 = n^2 - 20n + 100 + (n - 9) = (n - 10)^2 + (n - 9) > (n - 10)^2$,

And $n^2 - 19n + 91 = n^2 - 18n + 81 + (10 - n) < (n - 9)^2$,

So $(n - 10)^2 < n^2 - 19n + 91 < (n - 9)^2$, which implies that $n^2 - 19n + 91$ is not a perfect square.

(ii) When $n < 9$ then $n^2 - 19n + 91 = (10 - n)^2 + (n - 9) < (10 - n)^2$

and $n^2 - 19n + 91 = (9 - n)^2 + 10 - n > (9 - n)^2$,

So $(9 - n)^2 < n^2 - 19n + 91 < (10 - n)^2$ i.e, $n^2 - 19n + 91$ cannot be a perfect square.

(iii) when $n = 9$, then $n^2 - 19n + 91 = (10 - 9)^2 = 1$; when $n = 10$, then

$n^2 - 19n + 91 = (10 - 9)^2 = 1$

Thus, $n^2 - 19n + 91$ is a perfect square if and only if $n = 9$ or 10.

(M2) $n^2 - 19n + 91 = k^2$

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$4n^2 - 76n + 369 = 4k^2$
$(2n-19)^2 + 3 = 4k^2$
$(2k - 2n + 19)(2k + 2n - 19) = 3$

$2k - 2n + 19 = 3$	OR	$2k - 2n + 19 = 1$
$2k + 2n - 19 = 1$		$2k + 2n - 19 = 3$
$n = 9$		$n = 10$

Exercise – 7

1. In the following listed numbers, the one which must not be a perfect square is
   (a) $3n^2 - 3n + 3$ (b) $4n^2 + 4n + 4$
   (c) $5n^2 - 5n - 5$ (d) $7n^2 - 7n + 7$
   (e) $11n^2 + 11n - 11$
2. The last digit of the square of a natural number is 6. Prove that its next to last digit is odd.
3. The next to last digit of the square of a natural number is odd. Prove that its last digit is 6.
4. Is it possible to write a perfect square using only the digits (a) 2, 3, 6; (b) 1, 2, 3 exactly 10 times each?
5. Can the sum of the digit of a perfect square be equal to 1970?
6. Is there a three-digit number $\overline{abc}$ (where $a \neq c$) such that $\overline{abc} - \overline{cba}$ is a perfect square?
7. Determine if there is a natural number k such that the sum of the two numbers $3k^2 + 3k - 4$ and $7k^2 - 3k + 1$ is a perfect square.
8. If $(x-1)(x+3)(x-4)(x-8) + m$ is a perfect square, then m is
   (a) 32 (b) 24 (c) 98 (d) 196
9. If $n + 20$ and $n - 21$ are both perfect squares, where n is a natural number, find n.
10. Find the maximum integer x such that $4^{27} + 4^{1000} + 4^x$ is a perfect square.
11. Prove that there is no three-digit number $\overline{abc}$, such that $\overline{abc} + \overline{bca} + \overline{cab}$ is a perfect square.
12. Prove that the equation $a^2 + b^2 - 8c = 6$ has no integer solution.

1.8 Diophantine Equation

An equation with more than one variable for which integral solutions are required is called Diophantine equation. For example, $2x + 5y = 1$ or $x^2 - z^2 = y^3$, where x, y, and z are integers. Named in honour of the 3rd-century Greek mathematician Diophantus of Alexandria, these equations were first systematically solved by Hindu mathematicians beginning with Aryabhatta (c. 476–550).

Diophantine Equation Type 1

The simplest and typical equation is $ax + by = c$
where a, b, c are constant integers.
For example: $2x + 3y = 5$, $5x + 2y = 7$, $9x + 13y - 12 = 0$

Theorem

The equation "$ax + by = c$" has no integer solution if $\gcd(a, b) \nmid c$.

Proof. If (x, y) is an integer solution of "$ax + by = c$", then left hand side is divisible by gcd(a, b), whereas the right hand side is not, a contradiction.

If $\gcd(a, b) \nmid c$, then, after dividing both the sides by gcd(a, b), the equation "$ax + by = c$" becomes $a'x + b'y = c'$, where a', b', c' are integers and $\gcd(a', b') = 1$.

Theorem

If $\gcd(a, b) = h$, then there always exists $x, y \in Z$ such that $ax + by = h$.

Proof. By Euclid algorithm, we can write the HCF as linear combinations of the numbers.

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Theorem

If x_0, y_0 is a special integer solution of the equation $ax + by = c$, here $\gcd(a, b) = 1$, then the general solution of the equation is given by $\begin{cases} x = x_0 + bt \\ y = y_0 - at, \end{cases} \forall t \in Z$

Proof. Let (x, y) be any solution of equation different from the given solution (x_0, y_0). Then,

$ax + by = c$ (1)

$ax_0 + by_0 = c$ (2)

(1) - (2) gives $a(x - x_0) = -b(y - y_0)$, so $\frac{x - x_0}{b} = \frac{y - y_0}{-a}$.

Since $b \mid a(x - x_0)$ and $a \mid b(y - y_0)$ implies that $b \mid (x - x_0)$ and $a \mid (y - y_0)$, so each side of the last equality is an integer. Let it be t, it follows that

$x - x_0 = bt$ and $y - y_0 = -at$,

$x = x_0 + bt$ and $y = y_0 - at$, where $t \in Z$.

Note:

(i) For solving a diophantine equation of the form $ax + by = c$, we can find the general solution, if a special solution is given or can be found easily, then determine the range or values of t according to the given conditions in question.

(ii) For solving a system of diophantine equations or an equation with more than two variables, substitution method can be used for simplifying the equation.

Example 1: The number of integer solutions of the equation $2x + 4y = 21$ is ____.

Sol. Since gcd (2, 4) = 2,
And 2 does not divide 21.
So, the equation $2x + 4y = 21$ has no integer solution.

Example 2: Find the general solution of the Diophantine equation $x + 2y = 5$.

Sol. $x = 5 - 2y$
Let $y = k$, then $x = 5 - 2k$, where k is integer.

Example 3: Find all positive integer solutions to the equation $12x + 5y = 125$.

Sol. $12x = 5(25 - y)$ indicates $5 \mid x$. Let $x = 5$, then $5y = 65$ gives $y = 13$, so (5, 13) is a special solution.
Let $x = 5 + 5t$ and $y = 13 - 12t$; where t is an integer.
Since, $x \geq 1$, so $t \geq 0$. But $y \geq 1 \Rightarrow t \leq 1$, so, $t = 0$ or 1.
When $t = 0$, the solution is $x = 5$, $y = 13$.
When $t = 1$, then $x = 10$, $y = 1$.
Thus, the equation has exactly two solutions.

Example 4: Find the integer solutions of the equation $13x - 7y = 0$ satisfying the condition $80 < x + y < 120$.

Sol. Since $13x = 7y$ has integer solution (0,0) so the general solution is
$x = 7t$, $y = 13t$, $t \in \mathbb{Z}$
Then $x + y = 20t$, so $80 < x + y < 120 \Leftrightarrow 4 < t < 6$, i.e. $t = 5$. Thus, $x = 35$, $y = 65$ is the unique desired solution

Example 5: Find the general solution of the Diophantine equation $3x + 2y = 5$.

Sol. $y = (5 - 3x)/2 = 2 - x + (1 - x)/2$
Let $k = (1 - x)/2$
$x = 1 - 2k$

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$y = 2 - 1 + 2k + k = 3k + 1$, where k is an integer.

Example 6: Find the general solution of the Diophantine equation $3x + 6y = 5$.

Sol. $3x + 6y = 5$

$$x + 2y = \frac{5}{3}$$

Not possible as the left hand side is an integer and right hand side is a fraction.

Example 7: Given that the positive integers $x > 1$ and y satisfies the equation $2007x - 21y = 1923$. Find the minimum value of $2x + 3y$.

Sol. Simplify the given equation to $669x = 7y + 641$. Changing its form to $669(x-1) = 7(y-4)$ Since $x - 1 > 0$, so $y > 4$ and the minimum positive values of x and y are given by $x - 1 = 7$, $y - 4 = 669$. Thus. $2x_{min} + 3y_{min} = 2.8 + 3.673 = 2035$

.

Example 8: The digits a, b and c of a three digit number $\overline{abc}$ satisfy $49a + 7b + c = 286$. Find the three –digit number $\overline{abc}$.

Sol. By taking modulo 7 to both sides of the given equation, it follows that $c \equiv 6 \pmod 7$ Since c is a digit, so $c = 6$. Then the given equation becomes $7a + b = 40$ or $7a = 40 - b$. Since $31 \le 40 - b \le 40$, and there is only one number 35 divisible by 7 in this interval, so $b = 5$, $a = 5$, i.e. $\overline{abc} = 556$.

Example 9: Find all two-digit numbers such that each is divisible by the product of its two digits.

Sol. Let $\overline{xy} = 10x + y$ be a desired two-digit number. Then, there is a positive integer k such that $10x + y = kxy$; so $y = (ky - 10)x$, i.e. $x \mid y$. Thus, $0 < x \le y$. If $x = y$, then $11x = kx^2$, so $kx = 11$. i.e. $k = 11$, $x = 1 = y$. Thus, 11 is a solution. If $x < y$, then $x \le 4$ (otherwise, $y \ge 10$). When $x = 4$, then $y = 8$. However, there is no positive integer k such that $48 = 32k$, so $x \ne 4$. When $x = 3$, $10x = (kx - 1)y$ gives $30 = (3k - 1)y$. Since $y \le 9$, $3 \mid y$ and $y \mid 30$, so $y = 6$. It is obvious that $36 = 2.3.6$ so 36 is the second solution. When $x = 2$, then $20 = (2k - 1)y$. As $2k - 1$ is odd, so $y = 4$. $24 = 3.2.4$ verifies that 24 is the third solution. When $x = 1$ then $10 = (k - 1)y$. So, $y = 2$ or 5. $12 = 6.1.2$ and $15 = 3.1.5$ indicates that 12 and 15 are solutions as well. Thus, the complete solution set is {11, 12, 15, 24, 36}.

Diophantine Equation Type 2

All non-linear Diophantine equations comes in this category. For example, $x^2 + y^2 = z^2$, here x, y and z are integers. There are different approaches to solve these types of equations.

Basic Methods for Solving Quadratic Equations on integers(Z)

(i) Factorization Method.

Let the right hand side of the equation be a constant, zero or a power of a prime number (in the case of quadratic equation), and factorize the left hand side to the form of product of linear factors, then discuss the possible values of the linear factors based on the factorization of the right hand side.

(ii) Discriminant Method.

When a quadratic equation with integer coefficients has integer solution(s), its discriminant must be a perfect square. This feature will play an important role. When the quadratic equation contains two variables x, y, by the formula for roots, x can

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be expressed in terms of y, and its discriminant is an expression in y. Since the discriminant is a perfect square and that y is an integer, y can be found easily in many cases.

Note: Besides the use of discriminant, the use of **Vieta Theorem** and **transformation** or **substitution** are also useful tools for simplifying and solving quadratic equations.

(iii) Congruence, divisibility, and parity analysis, etc. are often used in discussing the existence of integer solutions of quadratic equations.

Example 10: Find integral solution (x, y) such that $xy = 6$.

Sol. x (or y) is a factor of 6
(6, 1), (1, 6), (3, 2), (2, 3), (-6, -1), (-1, -6), (-3, -2), (-2, -3)

Example 11: Find integral solution (x, y) such that $xy = x + y$.

Sol. $x + y - xy + 1 = 1$
$(x - 1)(y - 1) = 1$
$x - 1 = 1$ & $y - 1 = 1$ or $x - 1 = -1$ & $y - 1 = -1$
$x = 2, y = 2$ or $x = 0, y = 0$

Example 12: Find integral solutions (x, y) such that $(2x + y)(5x + 3y) = 7$.

Sol. $(2x + y)(5x + 3y) = 7$
(i) $2x + y = 7$ & $5x + 3y = 1$
$x = 20 ; y = -33$
(ii) $2x + y = 1$ & $5x + 3y = 7$
$x = -4 ; y = 9$
(iii) $2x + y = -7$ & $5x + 3y = -1$
$x = -20 ; y = 33$
(iv) $2x + y = -1$ & $5x + 3y = -7$
$x = 4 ; y = -9$

Example 13: Find integral solutions (x, y) such that $xy = x + y + 3$.

Sol. $xy - x - y + 1 = 4$
$\Rightarrow (x - 1)(y - 1) = 4$
$4 \rightarrow 1 \times 4, 4 \times 1, -1 \times -4, -4 \times -1, 2 \times 2, -2 \times -2$
$\Rightarrow (x, y) = (2, 5), (5, 2), (0, -3), (-3, 0), (3, 3), (-1, -1)$

Example 14: Find integral solutions (x, y) such that $x^2 = 14 + y^2$.

Sol. $x^2 - y^2 = 14$
$\Rightarrow (x - y)(x + y) = 14$
$14 \rightarrow 1 \times 14, 14 \times 1, -1 \times -14, -14 \times -1, 2 \times 7, 7 \times 2, -2 \times -7, -7 \times -2$
In L.H.S, either both factors are even or both factors are odd.
But in R.H.S, one factor is even & other odd.
$\Rightarrow$ No solution possible

Example 15: Find the number of ordered pairs of positive integers (x, y) that satisfy the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{2004}$.

Sol. By removing the denominators, it follows that $xy = 2004(x + y)$.

Then $xy - 2004x - 2004y + 2004^2 = 2004^2$, so $(x - 2004)(y - 2004) = 2004^2$

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For any positive factor p of 2004^2, $2004^2 = p.(2004^2/p)$ yields a solution $x = 2004 + p$, $y = 2004 + (2004^2/p)$. Since $2004^2 = (2^2 \times 3 \times 167)^2 = 2^4 \times 3^2 \times 167^2$, The number of positive divisors of 2004^2 is $(4 + 1)(2 + 1)(2 + 1) = 45$ so there are 45 such solutions. If p is a negative factor of 2004^2, then so is $2004^2/p$, and one of them must have absolute value not less than 2004, so one of $p + 2004$ and $(2004^2/p) + 2004$ is not positive, i.e., such solutions are not required. Thus, there are a total of 45 required solutions.

Exercise – 8

1. (CHINA/1990) An integer solution of the equation $1990x - 1989y = 1991$ is
   (a) $x = 12785; y = 12768;$ (b) $x = 12785; y = 12770;$
   (c) $x = 11936; y = 11941;$ (d) $x = 13827; y = 12623.$
2. (SSSMO(J)/2002) Two positive integers A and B satisfy $\frac{A}{11} + \frac{B}{3} = \frac{17}{33}$. Find the value of $A^2 + B^2$.
3. A dragonfly has six feet and a spider has 8 feet. Given that a certain group of dragonflies and spiders have in total 46 feet, find the number of dragonflies and the number of spiders.
4. If a four digit number and the sum of all its digits have a sum 2006, find the four digit number.
5. (Ancient Question) In an ancient chicken market, each rooster is sold for 5 coins, each hen for 3 coins and each chick for 1/3 coins. Someone has 100 coins to buy 100 chickens, how many roosters, hens and chicks can a man purchase out of a total cost of 100 coins?
6. Solve the equation $3x + 5y = 7$ in integers.
7. Solve the equation $1990x - 173y = 11$ in integers.
8. Find all integer roots of the equations $21x + 48y = 6$.
9. Find all integer roots of the equation $x^2 + y^2 = 4z - 1$
10. Solve for integer values of the variables: $x^3 + 21y^2 + 5 = 0$.
11. Solve for integer values of the variables: $15x^2 - 7y^2 = 9$.
12. Solve for integer values of the variables: $x^2 - y^2 = 1988$.
13. (CHINA/2003)) Find the integer solutions of the equation $6xy + 4x - 9y - 7 = 0$.
14. (KIEV/1962) Prove that the equation $x^2 + y^2 = 3z^2$ has no integer solution $(x; y; z) \neq (0; 0; 0)$.
15. (CHINA/2003) Given that $\frac{1260}{a^2 + a - 6}$ is a positive integer, where a is a positive integer. Find the value of a.
16. (CHINA/2001) How many number of pairs (x, y) of two integers satisfy the equation $x^2 - y^2 = 12$?
17. (SSSMO(J)/2004) Let x, y, z and w represent four distinct positive integers such that $x^2 - y^2 = z^2 - w^2 = 81$. Find the value of $xz + yw + xw + yz$.
18. (CHINA/2003) Find the number of non-zero integer solutions (x, y) to the equation $\frac{15}{x^2y} + \frac{3}{xy} - \frac{2}{x} = 2$.
19. CHINA/2001) Find the number of positive integer solutions to the equation $\frac{x}{3} + \frac{14}{y} = 3$.
20. (CHINA/2001) Find the number of positive integer solutions of the equation $\frac{2}{x} - \frac{3}{y} = \frac{1}{4}$.
21. (SSSMO/2003) Let p be a positive prime number such that the equation $x^2 - px - 580p = 0$ has two integer solutions. Find the value of p.
22. (CHINA/1993) The number of positive integer solutions (x, y, z) for the system of simultaneous equations $\begin{cases} xy + xz = 255 \\ xy + yz = 31 \end{cases}$ is:
    (a) 3; (b) 2; (c) 1; (d) 0

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1.9 Important Theorems:

Chinese Remainder Theorem:

Let $m_1, m_2, \ldots, m_r$ denote r positive integers that are relatively prime in pairs, and let $a_1, \ldots, a_r$ denote any r integers. Then the congruences $x \equiv a_i \pmod{m_i}$, $i = 1,\ldots,r$ have common solutions. Any two solutions are congruent modulo $m_1m_2\ldots m_r$

Proof.

Let $M = m_1m_2\ldots\ldots m_r$. Note that M/m_j and m_j are coprime. Hence there exists b_j such $(M/m_j)b_j \equiv 1 \pmod{m_j}$. Clearly

$$\frac{M}{m_j} b_j \equiv 0 \pmod{m_i} \text{ for } i \neq j$$

Define $x_0 = \sum_{j=1}^{r} \frac{M}{m_j} b_j . a_j$. Now, $\frac{M}{m_j} b_j \equiv 1$ or $0 \pmod{m_i}$ according as $j = i$ or $j \neq i$. Hence x_0 is a solution of the system of congruences. If x_0 and y_0 are solutions of the system then $x_0 - y_0 \equiv a_i - a_i \equiv 0 \pmod{m_i}$ for each i. As $m_1, \ldots, m_r$ are coprime in pairs, we get $x_0 - y_0 \equiv 0 \pmod{M}$ as required.

Example 1: Find all solutions of the system $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$

Sol. Here $m_1 = 3, m_2 = 5$,

$M = m_1m_2 = 15$,

$M_1 = M/m_1 = 5, M_2 = M/m_2 = 3$

$M_1 \equiv 2 \pmod 3$, hence $M_1' \equiv 2 \pmod 3$.

$M_2 \equiv 3 \pmod 5$, hence $M_2' \equiv 2 \pmod 5$.

$$x_0 = 2M_1M_1' + 3M_2M_2' = 2\times5\times2 + 3\times3\times2 = 38 \equiv 8 \pmod{15}$$

$x_0 = 8 + 15K, K \in Z$

(M2) $x = 3m + 2 \Rightarrow 2, 5, 8, 11, 14, \ldots\ldots$

$x = 5n + 3 \Rightarrow 3, 8, 13, 18, \ldots\ldots$

$x = 8 + k$ L.C.M (3,5)

$= 8 + 15k ; k \in Z$

Example 2: Find all solutions of the system $x \equiv 4 \pmod 5$, $x \equiv 6 \pmod 7$

Sol. Here $m_1 = 5, m_2 = 7$,

$M = m_1m_2 = 35$,

$M_1 = M/m_1 = 7, M_2 = M/m_2 = 5$

$M_1 \equiv 2 \pmod 5$, hence $M_1' \equiv 3 \pmod 5$.

$M_2 \equiv 5 \pmod 7$, hence $M_2' \equiv 3 \pmod 7$.

$$x_0 = 4M_1M_1' + 6M_2M_2' = 4\times7\times3 + 6\times5\times3 = 174 \equiv 34 \pmod{35}$$

$x_0 = 34 + 35K, K \in Z$

Example 3: Find all solutions of the system $x \equiv 2 \pmod 4$, $x \equiv 3 \pmod 5$, $x \equiv 1 \pmod 7$

Sol. Here $m_1 = 4, m_2 = 5, m_3 = 7, M = m_1m_2m_3 = 140, M_1 = M/m_1 = 35, M_2 = M/m_2 = 28, M_3 = M/m_3 = 20$.

$M_1 \equiv 3 \pmod 4$, hence $M_1' \equiv 3 \pmod 4$.

$M_2 \equiv 3 \pmod 5$, hence $M_2' \equiv 2 \pmod 5$.

$M_3 \equiv 6 \pmod 7$, hence $M_3' \equiv 6 \pmod 7$. Consider

$$x_0 = 2M_1M_1' + 3M_2M_2' + 1\,M_3M_3'$$

$$2\times35\times3 + 3\times28\times2 + 20\times6 = 498 \equiv 78 \pmod{140}$$

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Fermat's Theorem

Let p be prime number and A be a number not divisible by p. Then $A^{p-1} \equiv 1 \pmod{p}$

Proof: Consider the p – 1 numbers: A, 2A, 3A,....., (p–1)A. We can show that no two of these numbers have the same remainder when divided by p. Indeed, if $kA \equiv nA \pmod{p}$, then $k \equiv n \pmod{p}$. This is impossible if k and n are unequal and both are less than p. Therefore, among the remainders of these p – 1 numbers, when divided by p each of the numbers through p -1 is represented exactly once. Multiplying them together, we have A.2A.3A .... (p–1)A ≡1.2.3 ... (p – 1) (mod p). That is, $(p-1)!\, A^{p-1} \equiv (p-1)! \pmod{p}$. Now p is prime, which implies that (p –1)! and p are relatively prime. We can cancel (p – 1)! from both sides of the congruence to obtain $A^{p-1} \equiv 1 \pmod{p}$.

Corollary:

Let p be a prime number. Then for any integer A we have $A^p \equiv A \pmod{p}$. Fermat's little theorem is not just an unexpected and "neat" fact. It also provides a very powerful tool for solving many problems in arithmetic. Some of these are given below.

Example 4: Find the remainder when 2^{100} is divided by 101.

Sol. Since 101 is prime

$2^{101-1} \equiv 1 \bmod(101)$

Remainder is 1.

Example 5: Find the remainder when 3^{102} is divide by 101.

Sol. Since 101 is prime

$3^{101-1} \equiv 1 \bmod(101)$

$3^{102} \equiv 9 \bmod(101)$

Remainder is 9.

Example 6: Prove that $300^{3000} - 1$ divisible by 1001.

Sol. $1001 = 7 \times 11 \times 13$

$300^{6} \equiv 1 \bmod(7)$

$300^{3000} \equiv 1 \bmod(7)$

$300^{3000} - 1$ divisible by 7

Similarly, $300^{3000} - 1$ divisible by 11 and 13.

Therefore, $300^{3000} - 1$ divisible by 1001.

Euler's Totient Function

Euler's totient function denoted by $\Phi(n)$, is defined as the number of positive integers $\leq n$ that are relatively prime to n where 1 is counted as being relatively prime to all numbers.

For example, $\Phi(6) = 2$, As 1 and 5 are coprime to 6.

Results:

(i) For any prime number p, $\Phi(p) = p - 1$.

(ii) For any prime number p^k, $\Phi(p^k) = p^k(1 - 1/p)$.

(iii) $\Phi(1) = 1$.

(iv) If gcd(m, n) = 1 then $\Phi(mn) = \Phi(m)\,\Phi(n)$.

(v) If $n = p_1^{\alpha_1} \cdot p_1^{\alpha_2} \cdots p_k^{\alpha_k}$ then $\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)$

Example 7: Find $\Phi(20)$.

Sol. $20 = 2^2 \times 5$

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$\Phi(20) = 20(1 - 1/2)(1 - 1/5) = 8$

Lemma:

If (a, m) = 1 and $r_1, r_2, \ldots, r_{\phi(m)}$ are the positive integers less than m and relatively prime to m, then the remainders of $ar_1, ar_2, \ldots, ar_{\phi(m)}$ when divided by m are a permutation of $r_1, r_2, \ldots, r_{\phi(m)}$.

Proof: First we show that $ar_1, ar_2, \ldots, ar_{\phi(m)}$ ....(1) are all different and that they are all relatively prime to m. To show that they are all different, suppose that $ar_i \equiv ar_j \pmod m$ for some i and j ($1 \le i \le (m), 1 \le j \le (m)$). Since (a, m) = 1, we can cancel 'a' from both sides of the congruence to get $r_i \equiv r_j \pmod m$. Since r_i and r_j are less than m, it follows that $r_i = r_j$. Hence, $r_i \neq r_j$ implies $ar_i \not\equiv ar_j \pmod m$, and so the number in (1) are all different.

To prove that $ar_1, ar_2, \ldots, ar_{\phi(m)}$ are relatively prime to m, suppose that p is a prime common divisor of ar_i and m for some i, $1 \le i \le \phi(m)$. Since p is prime, either p | a or p | r_i. Thus, either p is a common divisor of a and m or of r_i and m. But (a, m) = (r_i, m) = 1, so both cases are impossible. Hence (ar_i, m) = 1 for each i, i = 1, 2, ..., $\phi(m)$.

Euler Theorem:

Let a, m be integers such that (a, m) = 1. Then $a^{\phi(m)} \equiv 1 \pmod m$

Proof: From above Lemma we know that $r_1r_2.....r_{\phi(m)} \equiv (ar_1)(ar_2) .... (ar_{\phi(m)}) \equiv a^{\phi(m)} (r_1r_2 .....r_{\phi(m)}) \pmod m$.

Since each of $r_1, r_2 ...., r_{\phi(m)}$ is relatively prime to m, it follows that their product is also; thus, that factor is cancelled in the last congruence, and we get $1 \equiv a^{\phi(m)} \pmod m$

Example 8: find the remainder when 97^{161} divided by 20.

Sol. (20=8) and (20, 97) = 1, $97^8 \equiv 1 \pmod{20}$

$97^{161} \equiv (97^8)^{20}97 \equiv 97 \equiv 17 \pmod{20}$, Remainder = 17

Example 9: Find the remainder when $3^{5^{11^{7!}}}$ is divided by 19.

Sol. By Fermat's little theorem we have $3^{18} \equiv 1 \pmod{19}$. This hints that we will want to find $5^{11^{7!}}$ in modulo 18. Now note that $\phi(18) = 6$. So, by Euler's theorem we have $5^6 \equiv 1 \pmod{18}$. This hints that we will want to evaluate $11^{7!}$ modulo 6, this is easy because $11^{7!} \equiv (-1)^{7!} \equiv 1 \pmod 6$. So, there exists 'a' such that $11^{7!} = 6a + 1$, hence $5^{11^{7!}} \equiv 5^{6a} \cdot 5 \equiv 5 \pmod{18}$, This means that there exists 'b' such that $5^{11^{7!}} = 18b + 5$, so, $3^{5^{11^{7!}}} \equiv 3^{18b} \cdot 3^5 \equiv 3^5 \equiv 15 \pmod{19}$.

Modular Multiplicative Inverse:

Integer 'a' is called the modular multiplicative inverse of 'b' under modulo m, if $ab \equiv 1 \pmod m$. For example: 3 and 4 are modular multiplicative inverse of each other under modulo 11.

Theorem:

The multiplicative inverse of "a modulo m" exists if and only if a and m are relatively prime (i.e., if gcd(a, m) = 1).

Proof: Case I: If gcd(a, m) = 1

By Euclid algorithm, we can write ax + my = 1, here x and y are integers.

If we take congruence modulo m, we get $ax \equiv 1 \pmod m$.

Case II: If $ax \equiv 1 \pmod m$

$\Rightarrow ax = km + 1$, here k is an integer

$\Rightarrow ax - km = 1$

$\Rightarrow$ let gcd(a, m) = d ≥ 1

$\Rightarrow d \mid 1$

$\Rightarrow d = 1$

Example 10: The following table shows the inverses of 1,....., 12 modulo 13.

Number	1	2	3	4	5	6	7	8	9	10	11	12
Inverse	1	7	9	10	8	11	2	5	3	4	6	12

Observe that $11! \equiv [(1)(1)][(2)(7)][(3)(9)][(4)(10)][(5)(8)]][(6)(11)] \equiv 1 \pmod{13}$ and hence $12! \equiv 12 \equiv -1 \pmod{13}$

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Wilson's Theorem:

Let p be a prime. Then $(p-1)! \equiv -1 \pmod{p}$

Proof: The result can easily be verified for $p = 2$ and $p = 3$.

We assume that $p \geq 5$. Now, given an integer i such that $1 \leq i \leq p-1$ there exists a unique j such that $ij \equiv ji \equiv 1 \pmod{p}$, $1 \leq j \leq p-1$. Moreover $i = j$ if and only if $i = 1$ or $p-1$. Hence $(p-2)! \equiv 1 \pmod{p}$. Hence, $(p-1)! \equiv p-1 \equiv -1 \bmod(p)$

Example 11: Find the remainder when 9! divided by 11.

Sol. By Wilson theorem $10! \equiv -1 \pmod{11}$

$10 \times 9! \equiv 10 \pmod{11}$

$9! \equiv 1 \pmod{11}$

Exercise – 9

1. Find the smallest multiple of 10 which has remainder 2 when divided by 3, and remainder 3 when divided by 7.
2. (CHINA/2002) When a positive integer n is divided by 5, 7, 9, 11, the remainders are 1, 2, 3, 4 respectively. Find the minimum value of n.
3. Prove that the number $30^{239} + 239^{30}$ is not prime.
4. Let p be a prime number and suppose a and b are arbitrary integers. Prove that $(a+b)^p \equiv a^p + b^p \pmod{p}$.
5. Let p and q be different primes. Prove that $p^q + q^p \equiv p + q \pmod{pq}$.
6. Let p and q be different primes, prove that $\left[\frac{p^q + q^p}{pq}\right]$ is even if $p, q \neq 2$, where [x] denotes the integral part of x.
7. Let p be prime, and suppose p does not divide some number a. Prove that there exists a natural number b such that $ab \equiv 1 \pmod{p}$.
8. Show that $\sum_{r=1}^{n-1} r(r)!$ is divisible by n if and only if n is prime.
9. Let n be a natural number not divisible by 17. Prove that either $n^8 + 1$ or $n^8 - 1$ is divisible by 17.
10. (a) Let p be a prime not equal to 3. Prove that the number 111....11 (p ones) is not divisible by p.
    (b) Let $p > 5$ be a prime. Prove that the number 1111.11 (p–1 ones) is divisible by p.
11. Show that, $11 \mid (5^{10} - 3^{10})$. More generally, if p is a prime such that $p \nmid a$ and $p \nmid b$, then show that $p \mid (a^{p-1} - b^{p-1})$.
12. Prove that (i) $7 \mid (111^{333} + 333^{111})$, and (ii) $39 \mid (53^{103} + 103^{53})$.
13. If p and q are distinct primes, show that $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$.
14. Find the remainder when $7^{200} + 11^{800}$ is divided by 101.
15. Show that $89 \mid (2^{44} - 1)$ and $97 \mid (2^{48} - 1)$

1.10 Base Systems

Number system is a mathematical presentation of numbers by using digits or other symbols in a consistent manner. This representation is unique for every number. The number of distinct symbols used in a particular number system is called the base of that number system. We all are familiar with Decimal Number System. In this we use 10 digits 0, 1, 2, ..., 9 to represent any number. Some commonly used number systems are

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Base	Name	Description
10	Decimal system	The most used system in the world, is used in arithmetic. Its ten digits are "0–9".
2	Binary numeral system	Used internally by nearly all computers The two digits are "0" and "1".
8	Octal system	This system is occasionally used in computing. The eight digits are "0–7".

If a number x written in number system with base y, then it is denoted by x_y. For example, $(18)_{10}$ represents the number eighteen in the decimal system; $(10)_2$ represents the number two in a base two (or binary) system.

Presentation of any number N in base b:

If $N = a_n \times b^n + \ldots + a_2 \times b^2 + a_1 \times b + a_0$ and $0 \le a_n, \ldots, a_2, a_1, a_0 \le b - 1$

Then $N = (a_n \ldots a_2 a_1 a_0)_b$

For example, $N = 2 \times 10^3 + 3 \times 10^2 + 1 \times 10 + 4$

$\Rightarrow N = 2314_{10}$

Ex. $N = 5 \times 8^3 + 3 \times 8^2 + 0 \times 8 + 4$

$N = 5304_8$

Conversion of one number system to another number system:

We can break this into three types of conversions

i. Any number system to decimal Number system

ii. Decimal system to any number system

iii. One number system to any other number system

i. Any number system to decimal Number system.

Let $N = (a_n \ldots a_2 a_1 a_0)_b = a_n \times b^n + \ldots + a_2 \times b^2 + a_1 \times b + a_0$

Just Calculate the value of the expression, that is the decimal form of N.

Ex. $N = 230_8$

Here base is 8

$N = 2 \times 8^2 + 3 \times 8 + 0 = 128 + 24 = 152$

ii. Decimal system to any number system:

Let us try to understand this with an example.

8	4320
8	540 – 0
8	67 – 4
8	8 – 3
	1 – 0

Ex. Convert 4320_{10} into octal system.

Step 1: As base is 8, we divide the number by 8 and write quotient and remainder below the number as shown. Now, we divide the quotient by 8 and write the new quotient and remainder below this. This process continues till we reach quotient value 1.

$4320_{10} = 540 \times 8 + 0$

$= (67 \times 8 + 4) \times 8 + 0 = 67 \times 8^2 + 4 \times 8 + 0$

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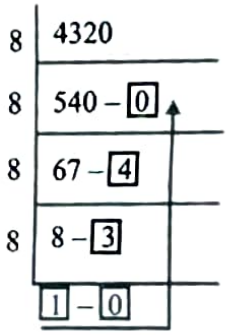
$= (8 \times 8 + 3) \times 8^2 + 4 \times 8 + 0 = 8 \times 8^3 + 3 \times 8^2 + 4 \times 8 + 0$
$= 8^4 + 3 \times 8^2 + 4 \times 8 + 0$
$= 10340_8$

Shortcut: Pick the marked digits, from bottom to top.



iii. One number system to any other number system:

In this first convert the number into decimal system then convert into desired number system.

Ex. Convert 1003200_4 into octal system.

$1003200_4 = 4^6 + 3 \times 4^3 + 2 \times 4^2 + 0 = 4320_{10}$
$4320_{10} = 10340_8$

Example 1: Convert 122_3 into binary system.

Sol. $122_3 = 1 \times 3^2 + 2 \times 3 + 2 = 17_{10}$
$17 = 2^4 + 0 \times 2^3 + 0 \times 2^2 + 0 \times 2 + 1 = 10001_2$

Example 2: Convert $(110101)_2$ into decimal system.

Sol. $(110101)_2 = 1*2^5 + 1*2^4 + 0*2^3 + 1*2^2 + 0*2^1 + 1*2^0 = (53)_{10}$

Example 3: How many 3-digit numbers in decimal system are 3-digit numbers in both base 9 and base 11?

Sol. Smallest 3-digit numbers in base 9 is
$100_9 = 1 * 9^2 + 0 * 9 + 0 * 9^0 = 81$
Largest 3-digit number in base 9 is
$888_9 = 8 * 9^2 + 8 * 9 + 8 * 9 = 728$
Smallest 3-digit number in base 11 is
$100_{11} = 1 * 11^2 + 0 * 11 + 0 * 11^0 = 121$
Largest 3-digit number in base 11 is
$101010_{11} = 10 * 11^2 + 10 * 11 + 10 * 11^0 = 1330$
So, 3-digit numbers in base 10 which are also 3-digit numbers in base 9 and base 11 are from 121 to $728 = 608$

Example 4: $(245)_x + (162)_x - (427)_x = 0$ in some base x. What is the value of x?

Sol. $(245)_x + (162)_x - (427)_x = 0$
$\Rightarrow (2x^2 + 4x + 5) + (x^2 + 6x + 2) - (4x^2 + 2x + 7) = 0$
$\Rightarrow x^2 - 8x = 0$
$\Rightarrow x = 8$

Example 5: A number N in base 10, is 503 in base b and 305 in base b + 2. What is the product of the digits of N?

Sol. $5b^2 + 3 = 3(b + 2)^2 + 5$
$2b^2 - 12b - 14 = 0$
$b^2 - 6b - 7 = 0$
$(b - 7)(b + 1) = 0$
$b = 7$ or $b = -1$
$N = 5 * 7^2 + 3 = 248$
Product of digits = 64

Example 6: In a number system $3 \times 5 = 23$, then the value of $1 \times 4 \times 5$ in the same number system is

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Sol. $3 \times 5 = 15_{10} = 23_b$

$15 = 2b + 3$

$b = 6$

$1 \times 4 \times 5 = 20_{10} = 3 \times 6 + 2 = 32$

Exercise – 10

1. Prove that $(5236)_9$ is divisible by 10.
2. The number $(2324a1013)_7$ is converted to base 10 and then divided by 6. It leaves no remainder. What could be the value of a?
   (a) 0 (b) 1 (c) 2 (d) 3
3. Find the fourth root of $(14641)_9$
4. How many 4-digit numbers in base 9 are perfect squares?
5. In a number system $4 \times 6 = 30$ and $5 \times 6 = 36$, then the value of $3 \times 4 \times 5$ in the same number system is
6. A two-digit number A in base 11 is one-third of the number formed by reversing its digits when considered in base 19. How many such numbers are possible?
7. In a number system the product of 44_b and 11_b is 1034_b. The number 3111_b of this system, when converted to the decimal number system, becomes
8. A three digit non-zero number 'ABC in base 5, when converted to base 7, becomes 'CBA'. Which of the following is necessarily true?
   (a) A must be 2 (b) B must be 0 (c) C must be 2 (d) None of these
9. Let a, b, c be distinct digits. Consider a two-digit number 'ab' and a three-digit number 'ccb', both defined under the usual decimal number system, if $(ab)^2 = ccb > 300$, then the value of b is
10. The product of two numbers 231 and ABA is BA4AA in a certain base system (where base is less than 10), where A and B are distinct digits. What is the base of that system?
11. What will be the number of zeros in $(2000!)_{14}$ Here 34 is the base in which the number is written.
12. In the familiar decimal number system, the base is 10, in another number system using base 4, the counting proceeds as 1, 2, 3, 10, 11, 12, 13, 20, 21, ... The twentieth number in this system will be
13. In a certain number system $363 + 1056 = 1452$. Find the value of $(654 - 456)$ in the same number system.
14. If $D = a^2 + b^2 + c^2$ where a and b are consecutive integers and $c = ab$, then $\sqrt{D}$ is:
   (a) Always an even integer
   (b) Always an odd integer
   (c) Sometimes an odd integer, sometimes not
   (d) Sometimes a rational number, sometimes not
15. If the number $(33333)^2 + 22222$ is expressed as a single decimal number, then the sum of its digits is

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Things to Remember

- **Prime:** A positive integer P is said to be prime, if
  (i) $P > 1$
  (ii) P has no divisors except 1 and P i.e. A number which has exactly two different factors, itself and one, is called prime number. Thus, 2, 3, 5, 7, 11, ... are primes.

Note: 2 is the only even number which is prime.

- **Composite:**
  A natural number n is said to be composite, if there exists integers l and m such that $n = rm$, where $1 < r < n$ and $1 < m < n$.

  Remark
- A prime number p can be written as a product only in one way namely p. 1
- A composite number has at least three factors.

Note: 1 is neither prime nor composite.

- **Twin Primes**
  A pair of numbers is sad to be twin primes. If they differ by 2.
  e.g., 3, 5 are twin primes.
- **Divisibility of Integers**
  An integer $x \neq 0$ divides y, if there exists an integer a such that $y = ax$ and thus we write as $x|y$ (x divides y). If x does not divide y, we write as $x \nmid y$ (x does not divide y)
  [This can also be stated as y is divisible by x or x is a divisor of y or y is a multiple of x].
- **Properties of Divisibility**
  1. $x|y$ and $y|z \Rightarrow x \mid z$
  2. $x|y$ and $x|z \Rightarrow x \mid (ky \pm lz)$ for all $k, l \in z$, z = set of all integers.
  3. $x|y$ and $y|x \Rightarrow x = \pm y$
  4. $x \mid y \Rightarrow x \mid yz$ for any integer z.
  5. $x \mid y$ iff $nx \mid ny$, where $n \neq 0$.
- **Number of Positive divisors of a Composite Number**
  If a composite number is $n = p_1^{\alpha_1} \times p_1^{\alpha_1} \times ..\times p_m^{\alpha_m}$, here $p_1, p_2, ..., p_m$ are primes. Then the number of positive divisors of n is $(\alpha_1 + 1)(\alpha_2 + 1)....(\alpha_m + 1)$.
- **Greatest Common Division**
  GCD of two natural numbers is the greatest natural number which divides them both.
  If d is GCD of a and b then
  (i) $d|a$ and $d|b$
  (ii) If $c \mid a$ and $c \mid b$, then $c \mid d$
  We write it as $(a, b) = d$
- **Properties of GCD**
  1. If $(b, c) = d$ and h is a common divisor of b and c, then h is a divisor of d.
  2. For any $m > 0$, $(mb, mc) = m (b, c)$
  3. If $d|b$ and $d|c$ and $d > 0$, then $\left(\frac{b}{d}, \frac{c}{d}\right) = \left(\frac{1}{d}\right)(b,c)$

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4. If (b, c) = d, then $\left(\frac{b}{d}, \frac{c}{d}\right) = 1$
5. If (a, b) = 1 and (a, c) = 1 then (a, bc) = 1
6. If a|bc and (a, b) = 1, then a | c
7. If a|b; then (a, b) = a
8. GCD of two coprime numbers is 1

Exponent of p in n!: If p is a prime number and e is the largest exponent of p such that $p^e | n!$, then $e = \sum_{i=1}^{\infty} \left[\frac{n}{p}\right]$

- **Test of Divisibility**

  Divisibility by certain special numbers can be determined without actually carrying out the process of division. The following theorem summarizes the result:

  A positive integer N is divisible by
  - 2 if and only if the last digit (unit's digit) is even.
  - 4 if and only if the number formed by last two digits is divisible by 4.
  - 8 if and only if the number formed by the last three digits is divisible by 8.
  - 3 if and only of the sum of all the digits is divisible by 3.
  - 9 if and only if the sum of all the digits is divisible by 9.
  - 5 if and only if the last digit is either 0 or 5.
  - 25 if and only if the number formed by the last two digits is divisible by 25
  - 125 if and only if the number formed by the last three digits is divisible by 125
  - 11 if and only if the difference between the sum of digits in the odd places (starting from right) and sum of the digits in the even places (starting from the right) is a multiple of 11.

- **Division Algorithm**

  For any two natural numbers a and b, there exists unique numbers q and r called respectively quotient and remainder, a = bq + r, where $0 \leq r < b$.

Perfect Number

A number n is said to be perfect if the sum of all divisors of n (including n) is equal to 2n. For example 28 is a perfect number because divisors of 28 are 1, 2, 4, 7, 14, 28.

Sum of divisors of n (28) = 1 + 2+ 4 + 7 + 14 + 28 = 56 = 2n

Coprime integers

Two numbers a and b are said to be coprime, if 1 is only common divisors of a and b.

i.e., if GCD of a and b = 1 i.e., if (a, b) = 1

e.g., (4, 5) = 1, (8, 9) = 1.

Theorem: If a = qb + r, then (a, b) = (b, r)

Corollary : If (a, b) = 1, then (b, r) = (a, b) = 1

i.e., if a is coprime to b, then r is coprime to b. Here r is remainder obtained on dividing a by b.

- **Congruences**

  If a and b are two integers and m is a positive integer, then a is said to be congruent to b modulo m, if m divides a – b denoted by m | (a – b). In notation form, we express it as $a \equiv b \bmod m$ or $a - b \equiv 0 \pmod m$.

Results:

1. $a \equiv b \pmod m$, then m | (a - b) or (a – b) is a multiple of m.
2. If m | a, then $a \equiv 0 \pmod m$
3. If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then $a \pm c \equiv b \pm d \pmod m$
4. If $a \equiv b \pmod m$, then $ac \equiv bc \pmod m$.

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But the converse is not true i.e., it is not always possible to cancel a common factor from a congruence.

5. If $a \equiv b \pmod m$, then $a^k \equiv b^k \pmod m$ for every positive integer k.
6. If $a \equiv b \pmod m$ and $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + .. + p_{n-1}x + p_n$ is an integer rational function of x with in coefficients, then $f(a) \equiv f(b) \pmod m$

Notes:

1. $(x + y) \mid (x^{2n+1} + y^{2n+1}) \forall n \in N_0, x + y \neq 0$
2. $(x - y) \mid (x^n - y^n) \forall n \in N, x - y \neq 0$

Chinese Remainder Theorem:

Let $m_1, m_2 ... m_r$ denote r positive integers that are relatively prime in pairs, and let $a_1, ..., a_r$ denote any r integers. The congruences $x \equiv a_i \pmod{m_i}$, $i = 1, ..., r$ have common solutions. Any two solutions are congruent modulo $m_1m_2 ... m_r$.

Fermat's Theorem:

Let p be prime number and A be a number not divisible by p. Then $A^{p-1} \equiv 1 \pmod p$

Euler's Totient Function:

Euler's totient function denoted by $\Phi(n)$, is defined as the number of positive integers $\leq n$ that are relatively prime to where 1 is counted as being relatively prime to all numbers.

If $n = p_1^{n_1} \cdot p_1^{n_1} \cdots p_k^{n_k}$ then $\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$

Euler Theorem:

Let a, m be integers such that $(a, m) = 1$. Then $a^{\Phi(m)} \equiv 1 \pmod m$

Wilson's theorem:

Let p be a prime. Then $(p - 1)! \equiv -1 \pmod p$

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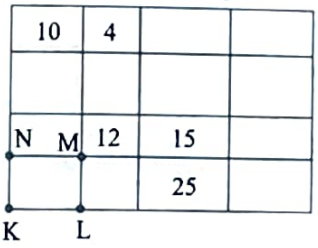
Practice Problems

1. [KVPY SA 2010] The sides of a triangle ABC are positive integers. The smallest side has length 1. Which of the following statement is true ?
   (a) The area of ABC is always a rational number
   (b) The area of ABC is always an irrational number
   (c) The perimeter of ABC is an even integer
   (d) The information provided is not sufficient to conclude any of the statements A, B or C above

2. [KVPY SA 2010] A rectangle is divided into 16 sub-rectangles as in the figure, the number in each sub rectangle represents the area of that sub-rectangle. What is the area of the rectangle KLMS ?



   (a) 20 (b) 30
   (c) 40 (d) 50

3. [KVPY SA 2011] The number of distinct prime divisors of the number 5123 – 2533 – 2593 is
   (a) 4 (b) 5
   (c) 6 (d) 7

4. [KVPY SA 2011] There are 30 questions in a multiple – choice test. A student gets 1 mark for each attempted question, 0 mark for each wrong answer and 4 marks for each correct answer. A student answered x question correctly and scored 60. Then the number of possible value of x is
   (a) 15 (b) 10
   (c) 6 (d) 5

5. [KVPY SA 2011] Let $f(x) = ax^2 + bx + c$, where a, b, c are integers. Suppose $f(1) = 0$, $40 < f(6) < 50$, $60 < f(7) < 70$, and $1000t < f(50) < 1000(t + 1)$ for some integer t. Then the value of t is
   (a) 2 (b) 3
   (c) 4 (d) 5 or more

6. [KVPY SA 2012] How many ordered pairs of (m, n) integers satisfy $\frac{m}{12} = \frac{12}{n}$?
   (a) 30 (b) 15
   (c) 12 (d) 10

7. [KVPY SA 2012] Let S = {1, 2, 3, ..., 40} and let A be a subset of S such that no two elements in A have their sum divisible by 5. What is the maximum number of elements possible in A ?
   (a) 10 (b) 13
   (c) 17 (d) 20

8. [KVPY SA 2012] Let a, b, c be positive integers such that $\frac{a\sqrt{2}+b}{b\sqrt{2}+c}$ is a rational number, then which of the following is always an integer ?
   (a) $\frac{2a^2+b^2}{2b^2+c^2}$ (b) $\frac{a^2+2b^2}{b^2+2c^2}$
   (c) $\frac{a^2+b^2-c^2}{a+b-c}$ (d) $\frac{a^2+b^2+c^2}{a+b-c}$

9. [KVPY SA 2012] The number of solutions (x, y, z) to the system of equations $x + 2y + 4z = 9$, $4yz + 2xz + xy = 13$, $xyz = 3$ Such that at least two of x, y, z are integers is –
   (a) 3 (b) 5
   (c) 6 (d) 4

10. [KVPY SA 2013] If a, b are natural numbers such that $2013 + a^2 = b^2$, then the minimum possible value of ab is
    (a) 671 (b) 668
    (c) 658 (d) 645

11. [KVPY SA 2013] How many natural numbers n are there such that n!+10 is a perfect square?
    (a) 1 (b) 2
    (c) 4 (d) infinitely many

12. [KVPY SA 2013] Let a, b, c, d, e be natural numbers in an arithmetic progression such that $a + b + c + d + e$ is the cube of an integer and $b + c + d$ is square of an integer. The least possible value of the number of digits of c is
    (a) 2 (b) 3
    (c) 4 (d) 5

13. [KVPY SA 2014] Let x and y be two 2- digit numbers such that y is obtained by reversing the digit of x. Suppose they also satisfy $x^2 - y^2 = m^2$ for some positive integer m. The value of $x + y + m$ is
    (a) 88 (b) 112
    (c) 144 (d) 154

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14. **[KVPY SA 2014]** The number of 6-digit numbers of the form ababab (in base 10) each of which is a product of exactly 6 distinct primes is-
(a) 8 (b) 10
(c) 13 (d) 15

15. **[KVPY SA 2014]** The houses on one side of a road are numbered using consecutive even numbers. The sum of the numbers of all the houses in that row is 170. If there are at least 6 houses in that row and a is the number of the sixth house, then-
(a) $2 \le a \le 6$ (b) $8 \le a \le 12$
(c) $14 \le a \le 20$ (d) $22 \le a \le 30$

16. **[KVPY SA 2014]** Suppose $a_2, a_3, a_4, a_5, a_6, a_7$ are integers such that $\frac{5}{7} = \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!} + \frac{a_6}{6!} + \frac{a_7}{7!}$ where $0 \le a_j < j$ for $j = 2, 3, 4, 5, 6, 7$. The sum $a_2 + a_3 + a_4 + a_5 + a_6 + a_7$ is-
(a) 8 (b) 9
(c) 10 (d) 11

17. **[KVPY SA 2014]** The number of all 3-digit numbers abc (in base 10) for which
$(a \times b \times c) + (a \times b) + (b \times c) + (c \times a) + a + b + c = 29$ is
(a) 6 (b) 10
(c) 14 (d) 18

18. **[KVPY SA 2015]** The largest non-negative integer k such that 24^k divides 13! is -
(a) 2 (b) 3
(c) 4 (d) 5

19. **[KVPY SA 2015]** Let $n > 1$ be an integer. Which of the following sets of numbers necessarily contains a multiple of 3
(a) $n^{15} - 1, n^{19} + 1$ (b) $n^{19}, n^{39} - 1$
(c) $n^{55}, n^{55} + 1$ (d) $n^{28}, n^{19} - 1$

20. **[KVPY SA 2015]** The number of distinct primes dividing $12! + 13! + 14!$ is -
(a) 5 (b) 6
(c) 7 (d) 8

21. **[KVPY SA 2016]** The number of digits in the decimal expansion of $16^{5^{16}}$
(a) 16 (b) 17
(c) 18 (d) 20

22. **[KVPY SA 2016]** Let t be real number such that $t^2 = at + b$ for some positive integers a and b. Then for any choice of positive integers a and b, t^3 is never equal to
(a) $4t + 3$ (b) $8t + 5$
(c) $10t + 3$ (d) $6t + 5$

23. **[KVPY SA 2016]** Consider a cuboid all of whose edges are integers and whose base is square. Suppose the sum of all its edges is numerically equal to the sum of the areas of all its six faces. Then the sum of all its edges is.
(a) 12 (b) 18
(c) 24 (d) 36

24. **[KVPY SA 2016]** The number of all 2-digit numbers n such that n is equal to the sum of the square of digit in its tens place and the cube of the digit in units place is
(a) 0 (b) 1
(c) 2 (d) 4

25. **[KVPY SA 2017]** The largest power of 2 that divides $\frac{200!}{100!}$
(a) 98 (b) 99
(c) 100 (d) 101

26. **[KVPY SA 2017]** Let S be the set of all ordered pairs (x, y) of positive integers satisfying the condition $x^2 - y^2 = 12345678$. Then
(a) S is an infinite set
(b) S is the empty set
(c) S has exactly one element
(d) S is a finite set and has at least two elements.

27. **[KVPY SA 2017]** If a, b, c, d are four distinct numbers chosen from the set {1,2,3 ....,9}, then the minimum value of $\frac{a}{b} + \frac{c}{d}$ is
(a) $\frac{3}{8}$ (b) 1/3
(c) 13/36 (d) 25/72

28. **[KVPY SA 2017]** It is given that the number 43361 can be written as a product of two distinct prime numbers p_1, p_2. Further, assume that there are 42900 numbers which are less than 43361 and are co-prime to it. Then, $p_1 + p_2$ is
(a) 462 (b) 464
(c) 400 (d) 402

29. **[KVPY SA 2018]** Let s be the sum of the digits of the number 152×518 in base 10. Then
(a) $s < 6$ (b) $6 \le s < 140$
(c) $140 \le s < 148$ (d) $s \ge 148$

30. **[KVPY SA 2018]** All the vertices of rectangle are of the form (a, b) with a, b integers satisfying the equation $(a - 8)^2 - (b - 7)^2 = 5$. Then the perimeter of the rectangle is
(a) 20 (b) 22
(c) 24 (d) 26

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31. [KVPY SA 2018] What is the sum of all natural number n such that the product of the digits of n (in base 10) is equal to $n^2 - 10n - 36$?
(a) 12 (b) 13
(c) 124 (d) 2612

32. [KVPY SA 2019] Let N be the least positive integer such that whenever a non-zero digit c is written after the last digit of N, the resulting number is divisible by c. Then sum of the digits of N is
(a) 9 (b) 18
(c) 27 (d) 36

33. [KVPY SA 2019] Let $N_1 = 2^{55} + 1$ and $N_2 = 165$. Then
(a) N_1 and N_2 are coprime
(b) the HCF (Highest Common Factor) of N_1 and N_2 is 55
(c) the HCF of N_1 and N_2 is 11
(d) the HCF of N_1 and N_2 is 33

34. [KVPY SA 2019] A prime number p is called special if there exist prime p_1, p_2, p_3, p_4 such that
$p = p_1 + p_2 = p_3 - p_4$.
The number of special primes is
(a) 0
(b) 1
(c) more than one but finite
(d) infinite

35. The sum of two positive integers is 52 and their LCM is 168. Find the numbers.

36. An infinite sequence of positive integers (a_n) is such that for any two positive integers $i \neq j$ $\gcd\{a_i, a_j\} = \gcd\{i, j\}$. Show that $a_i = i$ for all i.

37. The sum of the squares of three natural numbers is divisible by 9. Prove that we can choose two of these numbers such that their difference is divisible by 9

38. Three prime numbers p, q, and r, all greater than 3, form an arithmetic progression: $p, q = p + d$, and $r = p + 2d$. Prove that d is divisible by 6.

39. Prove that the number 100...00500...001 (100 zeros in each group) is not a perfect cube.

40. Prove that the number $6n^3 + 3$ cannot be a perfect sixth power of an integer for any natural number n.

41. If a nine-digit number is formed by the nine non-zero digits, and its unit digit is 5, prove that it must not be perfect square.

42. For any $n>1$ prove that the sum of any n consecutive odd natural numbers is a composite number.

43. Prove that if $(n-1)! + 1$ is divisible by n, then n is a prime number.

44. Prove that there exists a natural number n such that the numbers $n + 1, n+2, \ldots, n + 1989$ are all composite

45. Prove that there are infinitely many prime numbers.

46. (a) If it is known that $a + 1$ is divisible by 3, prove that $7a + 4$ is also divisible by 3.
(b) It is known that $2 + a$ and $35 - b$ are divisible by 11. Prove that $a + b$ is also divisible by 11

47. Find the last digit of the number 7^{7^7}

48. Find the last two digits of $14^{14^{14}}$

49. (SSSMO(J)/2001) Write down the last four digits of the number 7^{128}

50. If p and q are primes such that $p = q + 2$, prove that $p^p + q^q$ is a multiple of $p + q$

51. If a, b are integers, p is a prime, then show that $(a+b)^p \equiv a^p + b^p \pmod{p}$

52. Let 'a' be a rational number. Show that if $11 + 11\sqrt{11a^2 + 1}$ is an odd integer, then it must be a perfect square.

53. Prove that $2^p + 3^p$ is not a perfect power (i.e. a perfect square, cube etc.) if p is a prime number

54. Let $f(m, n) = 36^m - 5^n$, where m, n are natural numbers. Find the smallest value of $|f(m,n)|$. Justify your answer.

55. Prove that 10^{3n+1} cannot be represented as a sum of the cubes of two integers.

56. A three digit number was decreased by the sum of its digits. Then the same operation was carried out with the resulting number, etc, 100 times in all. Prove that the final number is zero

57. Let A be the sum of the digits of 4444^{4444}, and B the sum of the digits of A. Find the sum of the digits of B

58. Let a, b, c, d be distinct digits. Prove that $\overline{cdcdcdcd}$ is not divisible by $\overline{aabb}$

59. Prove that if p is a prime and a, b are any positive integers, then
(i) $\binom{ap}{p} \equiv 2 \pmod{p}$ (ii) $\binom{ap}{p} \equiv 2 \pmod{p^2}$.

60. (CHINA/1992) If x and y are positive integers, prove that the values of $x^2 + y + 1$ and $y^2 + 4x + 3$ cannot both be perfect squares at the same time.

61. (SSSMO(J)/1997) Suppose x, y and z are positive integers such that $x > y > z > 663$ and x, y and z satisfy the following:
$x + y + z = 1998$
$2x + 3y + 4z = 5992$:
Find the values of x, y and z.

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62. (ASUMO/1988) Prove that there are infinitely many positive integer solutions (x, y, z) to the equation $x - y + z = 1$, such that x, y, z are distinct, and any two of them have a product which is divisible by the remaining number.
63. Prove that for relatively prime two positive integers a and b, the equation $ax + by = c$ must have non-negative integer solution if $c > ab - a - b$.
64. Find integer solutions of $3 \times 2^m + 1 = n^2$
65. Find integer solutions (a, b), if $1/a + 1/b + 1/c = 1$
66. Find integer solutions (x, y), if $x^3 + 3 = 4y(y + 1)$
67. Find integer solutions (x, y), if $x^2 - 2y^2 = 1$(Pell's equation)
68. (USSR) Solve, in integers, $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$ (find the formula for general solution)
69. Find the last three digits of the 100^{th} powers of the first 100 natural numbers
70. (CHNMOL/2005) p, q are two integers, and the two roots of the equation in

$x : x^2-\frac{p^2+11}{9}x+\frac{15}{4}(p+q)+16=0$ are p and q also. Find the values of p and q.
71. (CHNMO(P)/2002) A positive integer is called a "good number" if it is equal to four times of the sum of its digits. Find the sum of all good numbers.
72. (ASUMO/1988) There are two piles of pebbles, pile (A) and pile (B). When 100 pebbles are moved from (A) to (B), then the number of pebbles in (B) is double of that in (A). However, if some are moved from (B) to (A), then the number of pebbles in (A) is five times more than that in (B). What is the minimum possible number of pebbles in (A), and find the number of pebbles in (B) in that case.
73. Show that if x and y are positive integers such that $x^2 + y^2 - x$ is divisible by 2xy, then x is a perfect square.
74. (i) Find the remainder of $\left(257^{33}+46\right)^{26}$ when it is divided by 50

    (ii) Find the remainder of $47^{37^{27}}$ when it is divided by 11.
75. Prove that the product of the last digit of the number 2^n and of the sum of all its digit but the last is divisible by 3.
76. Prove that for any positive integer n, $n^4 + 2n^3 + 2n^2 + 2n + 1$ is not a perfect square
77. Show that if $2n + 1$ and $3n + 1$ are both perfect squares then $40|n$.
78. Call a natural number n "convenient", if n^2+ divisible by 1000001. Prove that among number 1,2,.....1000000 there are evenly many "convenient" numbers.
79. The sum of the numbers a, b and c is divisible by 3. Prove that $a^5 + b^5 + c^5$ is also divisible by 30.
80. Prove that for each prime p the difference 111...11222...2233...33...88...88999...99 123456789 (in the first number each non-zero is written exactly p times) is divisible by p.
81. (USSR/1962) Prove that the only solution in integers of the equation $x^2+y^2+z^2=2xyz$ is $x=y=z=0$.
82. (MOSCOW/1940) Find all three-digit numbers such that each is equal to the sum of the factorials of its own digits.
83. (SSSMO/2005) How many ordered pairs of integers (x, y) satisfy the equation $x^2+y^2=2(x+y)+xy$?
84. (BMO/1991) Prove that the number $3^n + 2 \times 17$, where n is a non-negative integer, is never a perfect square.
85. (IMO/1986) Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set {2, 5, 13, d} such that $ab - 1$ is not a perfect square. Thus, in any case at least one of three numbers $2d - 12$, $5d - 1$, $13d - 1$ is not a perfect square
86. (KIEV/1980) Multiply some natural number by 2 and then plus 1, and then carry out this operation on the resultant number, and so on. After repeating 100 times of such operations, whether the resulting number is divisible by (i) 1980? (ii) 1981?
87. (AIME II 2012) For a positive integer p, define the positive integer n to be p-safe if n differs in absolute value by more than 2 from all multiples of p. For example, the set of 10-safe numbers is 3, 4, 5, 6, 7, 13, 14, 15, 16, 17,23 ..... Find the number of positive integers less than or equal to 10,000 which are simultaneously 7-safe, 11-safe, and 13-safe.

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88. (AIME/1986) In a parlor game, the magician asks one of the participants to think of a three digit number $\overline{abc}$ where a; b, and c represent digits in base 10 in the order indicated. The magician then asks this person to form the numbers $\overline{acb}, \overline{bca}, \overline{bac}, \overline{cab}$ and $\overline{cba}$ to add these five numbers, and to reveal their sum, N . If told the value of N , the magician can identify the original number, $\overline{abc}$. Play the role of the magician and determine the $\overline{abc}$ if N = 3194.
89. With each entry 1 submit, I have to write a different pair of positive integers whose greatest common factor is 1 and whose sum is 2000. (Pairs differing only in the order of addition are counted as 1 pair, NOT two different pairs) For example, I submitted the pair (1,1999) with my first entry. With these restrictions, at most how many entries can one person submit?
90. Let [r, s] denote the least common multiple of positive integers r and s. Find the number ordered triples (a, b, c) of positive integers for which [a, b] = 1000, [b, c] = 2000, and [c, a] =2000
91. Given natural numbers x, y. and z such that $x^2 + y^2 = z^2$. Prove that xy is divisible by 12.
92. (IMO/1994) Determine all ordered pairs (m; n) of positive integers such that $\frac{n^3+1}{mn-1}$ is an integer.
93. (USAMO/1975) Determine all integral solutions of $a^2+b^2+c^2=a^2b^2$
94. (IMO/Shortlist/1989) Given the equation
$4x^3+4x^2y-15xy^2-18y^3-12x^2$
$+6xy+36y^2+5x-10y=0$
Find all positive integer solutions.
95. (RMO 2016) For any natural number n, expressed in base 10, let S(n) denote the sum of all digits of n. Find all natural numbers n such that $n^3 = 8\,S(n)^3 + 6n\,S(n) + 1$.
96. (RMO 2016) How many 6-digit natural numbers containing only the digits 1, 2, 3 are there in which 3 occurs exactly twice and the number is divisible by 9?
97. (RMO 2015) Find all integers a, b, c such that $a^2 = bc + 1$, $b^2 = ca + 1$.
98. (RMO 2018)For a rational number r, its period is the length of the smallest repeating block in its decimal expansion. For example, the number r = 0.123123123..... has period 3. If S denotes the set of all rational number r of the form $r = \overline{abcdefgh}$ having period 8, find the sum of all the elements of S.
99. (RMO 2015) Find all three digit natural numbers of the form $(abc)_{10}$ such that $(abc)_{10}$, $(bca)_{10}$ and $(cab)_{10}$ are in geometric progression. (Here $(abc)_{10}$ is representation in base 10).

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Pre RMO PYQ's

1. **[IOQM 2021]** Consider the set of all 6-digit number consisting of only 3 digits, a,b,c, where a, b, c are distinct, suppose the sum of all these numbers is 593999406. What is the largest remainder when the three digit number abc is divided by 100.
2. **[IOQM 2020 KV]** Find the largest 2-digit number N which is divisible by 4, such that all integral power of N end with N.
3. **[IOQM 2020 KV]** The prime numbers a, b and c are such that $a + b^2 = 4c^2$. Determine the sum of all possible values of a + b + c.
4. **[IOQM 2020 KV]** If x and y are positive integers such that $(x - 4)(x - 10) = 2^y$, find the maximum possible value of x + y.
5. **[IOQM 2020 KV]** Find the largest positive integer N such that the number of integers in the set {1, 2, 3,...N} which are divisible by 3 is equal to the number of integers which are divisible by 5 or 7 (or both).
6. **[IOQM 2020 KV]** A five digit number $n = \overline{abcde}$ is such that when divided respectively by 2, 3, 4, 5, 6 the remainders are a, b, c, d, e. What is the remainder when n is divided by 100?
7. **[IOQM 2020 KV]** Let a, b, c be three distinct positive integers such that the sum of any two of them is a perfect square and having minimal sum a + b + c. Find this sum.
8. **[IOQM 2020 KV]** For a natural number n let n' denote the number obtained by deleting zero digits, if any. (For example, if n = 260, n' = 26; if n = 2020, n' = 22.) Find the number of 3-digit numbers n for which n' is a divisor of n, different from n.
9. **[IOQM 2020]** A number N in base 10, is 503 in base b and 305 in base b + 2. What is the product of the digits of N?
10. **[IOQM 2020]** What is the least positive integer by which $2^5 .3^6 .4^3 .5^3 .6^7$ should be multiplied so that, the product is a perfect square?
11. **[IOQM 2020]** A 5-digit number (in base 10) has digits k, k + 1, k + 2, 3k, k + 3 in that order, from left to right. If this number is m^2 for some natural number m, find the sum of the digits of m.
12. **[IOQM 2020]** Find they sum of all positive integers n for which $|2^n + 5^n - 65|$ is a perfect square
13. **[IOQM 2020]** The product $55 \times 60 \times 65$ is written as the product of five distinct positive integers. What is the least possible value of the largest of these integers?
14. **[IOQM 2020]** For a positive integer n, let $\langle n \rangle$ denote perfect square integer closest to n. For example $\langle 74 \rangle = 81, \langle 10 \rangle = 16$. If N is the smallest positive integer such that $\langle 91 \rangle . \langle 120 \rangle . \langle 143 \rangle . \langle 180 \rangle . \langle N \rangle = 91.120.143.180.N$ Find the sum of the square of the digits of N
15. **[IOQM 2020]** Positive integers a, b, c satisfy $\frac{ab}{a-b} = c$. What is the largest possible value of a + b + c not exceeding 99?
16. **[IOQM 2020]** Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number, a + 1 divides b − 1 and b divides $a^2 + a + 2$.
17. **[PRMO 25 Aug 2019]** Let N be the smallest positive such that $N + 2N + 3N + ... + 9N$ is a number whose all digits are equal. What is the sum of the digits of N?
18. **[PRMO 25 Aug 2019]** Let s(n) denote the sum of the digits of a positive integers n in base 10. If s(m) = 20 and s(33m) = 120, what is the value of s(3m)?
19. **[PRMO 25 Aug 2019]** Let $F_k(a,b) = (a + b)^k - a^k - b^k$ and let S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. For how many ordered pairs (a, b) with a, b ∈ S and $a \leq b$ is $\frac{F_5(a,b)}{F_3(a,b)}$ an integers?
20. **[PRMO 25 Aug 2019]** Find the largest value of a^b such that the positive integers a, b > 1 satisfy $a^b b^a + a^b + b^a = 5329$.
21. **[PRMO 25 Aug 2019]** Consider the sequence 1, 7, 8, 49, 50, 56, 57. 343,.... Which consists of sums of distinct powers of 7, that is $7^0, 7^1, 7^0 + 7^1, 7^2$, ..., in increasing order. At what position will 16856 occur in this sequence?
22. **[PRMO 25 Aug 2019]** In base − 2 notation, digits are 0 and 1 only and the places go up in powers of -2. For example, 11011 stands for $(-2)^4 + (-2)^3 + (-2)^1 + (-2)^0$ and equals numbers 7 in base 10. If the decimal number 2019 is expressed in base -2, how many non zero digits does it contain?

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23. [PRMO 25 Aug 2019] Let N denote the number of all natural numbers n such that n is divisible by a prime $p > \sqrt{n}$ and $p < 20$. What is the value of N?
24. [PRMO 25 Aug 2019] Let a, b, c, be distinct positive integers such that $b + c - a$, $c + a - b$ and $a + b - c$ are all perfect squares. What is the largest possible value of $a + b + c$ smaller than 100?
25. [PRMO 25 Aug 2019] Consider the set E of the positive integers n such that when divided by 9, 10, 11 respectively, the remainders (in that order) are all > 1 and form a non-constant geometric progression. If N is the largest element of E, find the sum of digits of E.
26. [PRMO 25 Aug 2019] For $n \geq 1$, let a_n be the number beginning with n 9's followed by 744; e.g., $a_4 = 9999744$. Define $f(n) = \max \{m \in N \mid 2^m \text{ divides } a_n\}$, for $n \geq 1$. Find $f(1) + f(2) + f(3) + \ldots\ldots + f(10)$.
27. [PRMO 16 Aug 2019] Let $\overline{abc}$ be a three digit number with nonzero digits such that $a^2 + b^2 = c^2$. What is the largest possible prime factor of $\overline{abc}$?
28. [PRMO 16 Aug 2019] Find the smallest positive integer $n \geq 10$ such that $n + 6$ is a prime and $9n + 7$ is a perfect square.
29. [PRMO 16 Aug 2019] How many ordered pairs (a, b) of positive integers with $a < b$ and $100 \leq a, b \leq 1000$ satisfy gcd (a, b) : lcm (a, b) = 1 : 495.
30. [PRMO 16 Aug 2019] Consider the set E of all natural numbers n such that when divided by 11, 12, 13 respectively, the remainders, in the order, are distinct prime numbers in an arithmetic progression. If N is the largest number in E, find the sum of digits of N.
31. [PRMO 16 Aug 2019] Consider the set E = {5, 6, 7, 8, 9}. For partition {A, B} of E, with both A and B non empty. Consider the number obtained by adding the product of elements of A to the product of elements of B. Let N be largest obtained by adding the product of elements of A to the product of elements of B. Let N be the largest prime number among these numbers. Find the sum of digits of N.
32. [PRMO 2018] Consider all 6-digit numbers of the form abccba where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7.
33. [PRMO 2018] The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$ (that is 1,6,5,8,9,0 are digits in base b in the above equation). Find the sum of all possible values of $b \geq 10$ satisfying the equation.
34. [PRMO 2018] Let a and b natural numbers such that $2a - b$, $a - 2b$ and $a + b$ are all distinct squares. What is the smallest possible value of b?
35. [PRMO 2018] If a, b, $c \geq 4$ are integers, not all equal, and $4abc = (a + 3)(b + 3)(c + 3)$, then what is the value of $a + b + c$?
36. [PRMO 2018] Determine the sum of all possible positive integers, the product of whose digits equals $n^2 - 15n - 27$.
37. [PRMO 2018] Let T be the smallest positive integers which, when divided by 11, 13, 15 leaves remainder in the sets {7,8,9}, {1,2,3}, {4,5,6} respectively. What is the sum of the square of the digits of T?
38. [PRMO 2017] How many positive integers less than 1000 have the property than the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3?
39. [PRMO 2017] A pen costs Rs 11 and a notebook cost Rs13. Find the number of ways in which a person can spend exactly Rs 1000 to buy pens and notebooks.
40. [PRMO 2017] Suppose an integer x, a natural n and a prime number p satisfy the equation $7x^2 - 44x + 12 = p^n$. Find the largest value of p.
41. [PRMO 2017] Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of 3pq for all positive integers n. Find the least possible value of $p + q$.
42. [PRMO 2017] For each positive integer n, consider the highest common factor h_n of the two numbers $n! + 1$ and $(n + 1)!$. For $n < 100$, find the largest value of h_n.
43. [PRMO 2015] Positive integers a and b are such that $a + b = a/b + b/a$. What is the value of $a^2 + b^2$?
44. [PRMO 2015] How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square?
45. [PRMO 2015] Let n be the largest integer that is the product of exactly 3 distinct prime numbers, x, y and $10x + y$, where x and y are digits. What is the sum of the digits of n?
46. [PRMO 2015] The digits of a positive integer n are four consecutive integers in decreasing order when read from left to right. What is the sum of the possible remainders when n is divided by 37?
47. [PRMO 2014] A natural number k is such that $k^2 < 2014 < (k + 1)^2$. What is the largest prime factor of k?

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48. [PRMO 2014] For natural numbers x and y, let (x, y) denote the greatest common divisor of x and y. How many pairs of natural numbers x and y with $x \le y$ satisfy the equation $xy = x + y + (x, y)$?

49. [PRMO 2014] For how many natural n between 1 and 2014 (both inclusive) is $\frac{8n}{9999-n}$ an integer?

50. [PRMO 2013] What is the smallest positive integer k such that $k(3^3 + 4^3 + 5^3) = a^n$ for some positive integers a and n, with $n > 1$?

51. [PRMO 2013] Let S(M) denote the sum of the of a positive integer M written in base 10. Le the smallest positive such that $S(N) = 2013$. is the value of $S(5N + 2013)$?

52. [PRMO 2012] For how many pairs of integers (x, y) is $x + 3y = 100$?

53. [PRMO 2012] Let $S_n = n^2 + 20n + 12$, n is a integer. What is the sum of all possible va for which S_n is a perfect square?

54. [PRMO 2012] How many integer pairs (x, y) $x^2 + 4y^2 - 2xy - 2x - 4y - 8 = 0$?

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1. Number Theory

Exercise-1:

1. Yes 2. No 3. (d) 4. 12 5. 60 6. $x = 52, y = 49$ or $x = 152, y = 151$ 7. (a)
8. (a) $P - 1$ (b) $P^2 - P$ 10. 45

Exercise -2:

5. (a) $p = 3$, (b) $p = 3$ 9. $x = 22 + 30k$; $K \in Z$ 10. $p = 3$

Exercise -3:

1. 7 2. (e) 3. (b) 4. 1 5. (c) 6. (d) 7. (c) 8. 83

Exercise -4:

1. (e) 2. $(x, y) = (64, 8), (8, 64)$ 3. 104, 96 4. $\begin{cases} 3, \text{if } n = 3k+1 \\ 1, \text{if } n \neq 3k+1 \end{cases}$ 5. 2 6. 9
7. 12 8. 2 9. 85

Exercise -5:

7. 4 8. 1 9. 1 10. (i) 6 (ii) 2 11. (a) $n = 3k, k \in Z$ (b) No n possible 12. 88
14. Not Possible

Exercise -6:

1. a 3. 111.....11599125 4. 2 5. 111...311 (three hundred 1's in the representation)

Exercise -7:

1. (b) 4. (a) not possible (b) not possible 5. No 6. No 7. Not possible 8. (d)
9. 421 10. 1972

Exercise -8:

1. C 2. 5 3. 5 dragonflies and 2 spiders or one dragonfly and 5 spiders 4. 1984, 2002
5. (0, 25, 75), (4, 18, 78), (8, 11, 81), (12, 4, 84) 6. $x = -1 + 5k, y = 2 - 3k$ where k is any integer
7. $x = x_0 + 173k = 22 + 173k$, $y = y_0 + 1990k = 253 + 1990k$, where k is any integer
8. $x = 16k - 2, y = -7k + 1$ 9. No Solution 10. No Solution 11. No Solution
12. $x = \pm 498, y = \pm 496$ and $x = \pm 78, y = \pm 64$ 13. $x = 1, y = -1$ 15. $a = 3, 4, 6, 9, 11$
16. (4, 2): (4, -2); (-4, -2) and (-4, 2) 17. 2187 18. $x = 3, y = 1$ 19. 5 20. 5
21. $p = 29$ 22. B

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Exercise -9:

1. (8) 2. 1731 14. 2

Exercise -10:

2. (c) 3. (116) 4. (54) 5. (74) 6. (5) 7. (406) 8. (b) 9. (1) 10. (6) 11. (123)
12. (110) 13. (165) 14. (b) 15. (10)

Practice Problems

1. (b) 2. (d) 3. (c) 4. (c) 5. (c) 6. (a) 7. (c) 8. (d) 9. (b) 10. (c)
11. (a) 12. (b) 13. (d) 14. (c) 15. (c) 16. (b) 17. (c) 18. (b) 19. (a) 20. (a)
21. (c) 22. (b) 23. (c) 24. (c) 25. (c) 26. (b) 27. (d) 28. (a) 29. (b) 30. (a)
31. (b) 32. (a) 33. (d) 34. (b) 35. 24, 28 47. 3 48. 36 49. 6801 54. 11 57. 7

61. $x = 667, y = 666, z = 665$

64. $n = 2, m = 1; n = 5, m = 3; n = 7, m = 4$

65. (1, a, −a), (b, 1, −b), (c, −c, 1), where a, b and c arbitrary integers. (1, 2, 3), (2, 4, 4), (3, 3, 3)

66. No Solution 67. (3a + 4b, 2a + 3b)

68. $x = m(m+n)t, y = \pm n(m+n)t, z = mnt$, where m, n, t are arbitrary integers

69. 000,001,625 and 376 70. (p; q) is (13; 7).

71. 120 72. 170, 40 74. 29, 5 82. 145 83. 6 86. No, Yes 87. 958 88. 358 89. 400 90. 70

92. (2,2), (2,1) (3,1), (5,2), (5,3), (1,2), (1,3) (2,5), (3,5)

93. $a = b = c = 0$ 94. $\{(1,1)\} \cup \{(k,2k) : \forall\, k \in \mathbb{N}\}$ 95. 17

96. Zero 97. (a, b, c) = (1, −1, 0), (−1,1, 0), (1,0, -1), (−1,0, 1), (0,1, -1), (0, −1, 1)

98. 49995000 99. 111, 222, 333, 444, 555, 666, 777, 888, 999, 432,864.

Pre RMO PYQ

1. (98) 2. (76) 3. (31) 4. (16) 5. (65) 6. (11) 7. (55) 8. (93) 9. (64) 10. (15)
11. (15) 12. (6) 13. (20) 14. (56) 15. (99) 16. (16) 17. (37) 18. (60) 19. (22) 20. (81)
21. (36) 22. (06) 23. (69) 24. (91) 25. Bonus 26. (75) 27. (29) 28. (53) 29. (20) 30. Bonus
31. (17) 32. (70) 33. (12) 34. (21) 35. (16) 36. (17) 37. (81) 38. (28) 39. (7) 40. (5)
41. (28) 42. (97) 43. (2) 44. (8) 45. (12) 46. (217) 47. (11) 48. (3) 49. (1) 50. (1)
51. (18) 52. (33) 53. (16) 54. (6)

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You can Refer to Google for Proper Solution

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